FORMULAS FOR TRACES FOR A SINGULAR STURM-LIOUVILLE DIFFERENTIAL OPERATOR

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I. M. Gel'fand, B. M. Levitan, and L. A. Dikiĭ have obtained some identities for the eigenvalues of a regular Sturm-Liouville operator (see reference [1]). These identities can be interpreted as an expression of the regularized spectral traces of the integer powers of an operator in terms of the operator itself. Such a formulation of the problem permits one to state a similar problem for an operator with a continuous spectrum.

In the present work, formulas are obtained which express some "spectral characteristics" of an operator

$$Ly \equiv -y'' + q(x)y, \quad 0 \leq x < \infty, \quad y(0) = 0$$

in terms of $q(x)$. These expressions are analogous to the identities for the eigenvalues. The method of proof is based on an investigation of the properties of the denominator of the resolvant. This method appears to be convenient also for the case of a regular operator.

1. In sequel we shall always assume that*

$$\int_0^\infty x\,|\,q(x)\,|\,dx < \infty.$$

The spectrum of the operator L consists of the interval $[0, \infty]$ —the continuous part—and of the finite number of the negative eigenvalues $\lambda_l = -\kappa_l^2\,(\kappa_l > 0;\ l = 1, 2, \cdots, m)$. Connected with the operator L, the following function is often considered:

$$M(s) = 1 + \int_0^\infty e^{isx} q(x)\,\phi(x, s)\,dx = A(s)\,e^{i\eta(s)},$$

$$(s = \sigma + i\tau,\ 0 \leq \tau < \infty,\ -\infty < \sigma < \infty).$$

The function $\eta(\sigma)$ —which is the argument of $M(\sigma)$ —is the so-called limiting phase.

Let R_λ be the resolvent of L. The operator, corresponding to $q(x) \equiv 0$ will be denoted with a zero at the top.

Theorem 1. *The operator* $R_\lambda - R_\lambda^0$ *has a trace*** *for* $\arg\lambda \neq 0$ *and* $\lambda \neq \lambda_l$ $(l = 1, 2, \cdots, m)$, $\operatorname{tr}(R_\lambda - R_\lambda^0) = -\frac{d}{d\lambda}\ln M(\sqrt{\lambda});\ 0 \leq \arg\sqrt{\lambda} \leq \pi$.

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*For the notations we use in this paper and for the properties of the operator L, we refer the reader to [2].

**The notion of trace of an abstract operator is described in [3].

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Corollary. $M(\sqrt{\lambda}) = \det(E + qR_\lambda^0)$.

For $q(x) \in L[0, \infty]$ we have

$$\ln M(\sqrt{\lambda}) = \frac{1}{\pi}\int_0^\infty \frac{\eta(\sqrt{z})}{z-\lambda}\,dz + \sum_{l=1}^m \ln\frac{\lambda-\lambda_l}{\lambda} \quad (\operatorname{Im}\sqrt{\lambda} > 0);$$

thus

$$\operatorname{tr}(R_\lambda - R_\lambda^0) = -\int_{-\infty}^{\infty} \xi(t)\, d\frac{1}{t-\lambda}, \tag{α}$$

where

$$\xi(t) = \begin{cases} \frac{1}{\pi}\eta(\sqrt{t}), & t > 0, \\ -\int_{-\infty}^{t} \sum_l \delta(z-\lambda_l)\,dz, & t < 0. \end{cases}$$

The formula of the type (α) have been considered in the work of I. M. Lifšic [4] in connection with another problem. They have been investigated in detail for abstract operators by M. G. Kreĭn [5]. In our example the connection of the function $\xi(t)$ with the limiting phase $\eta(k)$ is of interest.

2. In order to obtain the expressions similar to the identities for the eigenvalues, the following lemmas are used:

Lemma 1. *If $q(x) \in L[0, \infty]$, then for $0 < \operatorname{Re} z < 1/2$*

$$\frac{\pi}{2z}\sum_{l=1}^m \varkappa_l^{2z} = \sin\pi z \cdot L(z) - \cos\pi z \cdot H(z),$$

where

$$H(z) = \int_0^\infty k^{2z-1}\eta(k)\,dk, \quad L(z) = \int_0^\infty k^{2z-1}\ln A(k)\,dk.$$

Lemma 1 is a corollary of the analytic properties of the function $M(s)$ in the upper half-plane. It can be obtained by the contour integration of the function $\frac{d}{ds}M(s)\frac{1}{M(s)}s^{2z}$.

Lemma 2. *Suppose that for $x \geq 0$ $q(x)$ has a continuous derivative $q^{(n)}(x)$ $(n \geq 1)$, while $q^{(l)}(x)$ $(l = 0, \cdots, n)$ have finite limits for $x \to \infty$. Then, uniformly in the upper half-plane, the following asymptotic formulas are valid:*

$$M(s) \underset{|s|\to\infty}{=} 1 - \sum_{l=0}^n \frac{(-1)^{l+1}}{(2is)^{l+1}} V_l + o\left(\frac{1}{|s|^{n+1}}\right),$$

$$\ln M(s) \underset{|s|\to\infty}{=} -\sum_{p=1}^{n+1}\frac{(-1)^p}{(2is)^p} Q_p + o\left(\frac{1}{|s|^{n+1}}\right),$$

hence on the real axis:

$$\ln A(k) \underset{|k|\to\infty}{=} -\sum_{\mu=1}^{\left[\frac{n+1}{2}\right]}\frac{(-1)^\mu}{(2k)^{2\mu}} Q_{2\mu} + o\left(\frac{1}{|k|^{n+1}}\right),$$

$$\eta(k) \underset{|k|\to\infty}{=} -\sum_{\mu=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^\mu}{(2k)^{2\mu+1}} Q_{2\mu+1} + o\left(\frac{1}{|k|^{n+1}}\right).$$

In these formulas $V_l \equiv \lim_{\alpha\to\infty} V_l(\alpha)$, $V_l(\alpha)$ $(0 \leq \alpha)$ satisfy the following recurrent relations:

$$V_0(\alpha) = -\int_0^\alpha q(z)\,dz,$$

$$V_l(\alpha) = q^{(l-1)}(0) + \sum_{m=0}^{l-1} C_{l-1}^m \int_0^\alpha dz\, V_m(z)\, q^{(l-m-1)}(z) \qquad (l = 1, \ldots, n+1),$$

$$Q_p = V_{p-1} + \sum_{j=1}^{p-1} \frac{j}{p} V_{p-j-1} Q_j.$$

This lemma can be proved, by expressing $\phi(x, s)$ as a function of s in terms of the operator of transformation, substituting this in the expression of $M(s)$ and using the integral equation for the operator of transformation.

The asymptotic formulas for $\ln A(k)$ and $\eta(k)$ permit one to investigate the analytic continuation of $H(z)$ and $L(z)$ starting from the strip $0 < \operatorname{Re} z < 1/2$ to the right, using the usual methods (see, e.g., [6]). It then turns out that $H(z)$ has its simple poles at the points $1/2, 3/2, \cdots$, and $L(z)$ has its simple poles at the points $1, 2, \cdots$. The residue at these poles is directly expressible in terms of Q_μ. From here, as a result of the analytic continuation of the identities in Lemma 1, it follows:

Theorem 2. *Under the assumptions of Lemma 2 the following formulas are valid:*

$$(-1)^\mu \sum_{l=1}^m \varkappa_l^{2\mu} + \frac{2\mu}{\pi}\int_0^\infty k^{2\mu-1}\left[\eta(k) - \sum_{l=0}^{\mu-1} \frac{(-1)^{l+1}}{(2k)^{2l+1}} Q_{2l+1}\right] dk =$$
$$= (-1)^\mu \frac{\mu}{2^{2\mu}} Q_{2\mu} \quad \left(\mu = 1, 2, \ldots \leqslant \frac{n}{2}\right);$$

$$(-1)^\mu \sum_{l=1}^m \varkappa_l^{2\mu+1} - \frac{2\mu+1}{\pi}\int_0^\infty k^{2\mu}\left[\ln A(k) - \sum_{l=1}^{\mu} \frac{(-1)^{l+1}}{(2k)^{2l}} Q_{2l}\right] dk =$$
$$= (-1)^\mu \frac{2\mu+1}{2^{2\mu+2}} Q_{2\mu+1} \quad \left(\mu = 0, \ldots \leqslant \frac{n-1}{2}\right).$$

The first sequence of these formulas is analogous to the identities for the eigenvalues, while the second expresses the same relations in terms of the function $A(k)$.

Using Lemma 2, we obtain

$$Q_1 = -\int_0^\infty q(z)\,dz, \quad Q_2 = q(0),$$

$$Q_3 = q'(0) + \int_0^\infty q^2(z)\,dz, \quad Q_4 = q''(0) - 2q^2(0).$$

For $\mu = 1, 2$ we obtain from the first sequence of these formulas:

$$-\sum_{l=1}^m \varkappa_l^2 + \frac{2}{\pi}\int_0^\infty t\left[\eta(t) - \frac{1}{2t}\int_0^\infty q(z)\,dz\right] dt = -\frac{1}{4} q(0) \quad (\mu = 1).$$

This relation was already obtained in the work of one of the present authors [7].

$$\sum_{l=1}^m \varkappa_l^4 + \frac{4}{\pi}\int_0^\infty t^3\left[\eta(t) - \frac{1}{2t}\int_0^\infty q(z)\,dz - \left(\frac{1}{2t}\right)^3\left(q'(0) + \int_0^\infty q^2(z)\,dz\right)\right] dt = \frac{1}{8}\left(q''(0) - 2q^2(0)\right) \quad (\mu = 2).$$

The comparison of our results with the results obtained by L. A. Dikiĭ for the case of the finite interval shows the complete analogy of the results.*

*See equation (6.3) and (6.4) of [1]. Observe that in [1] the derivatives of $q(x)$ of the odd orders are considered to be 0 at the ends of the intervals.

3. We shall make several remarks for the case of a finite interval:

$$ly \equiv -y'' + p(x)y, \quad 0 \le x \le \pi; \quad y(0) = y(\pi) = 0; \quad r_\lambda = (l - \lambda)^{-1}.$$

Here the denominator of the resolvent can be connected with an entire function $\omega(\lambda) \equiv \omega(\pi, \lambda)$, where

$$-\omega''(x, \lambda) + p(x)\omega(x, \lambda) = \lambda\omega(x, \lambda), \quad \omega(0, \lambda) = 0, \quad \omega'(0, \lambda) = 1,$$

Outside of the spectral points we have

$$\operatorname{tr} r_\lambda = -\frac{d}{d\lambda} \ln \omega(\lambda).$$

The representation of $\omega(\lambda)$ is given by

$$\omega(\lambda) = \frac{\sin \sqrt{\lambda}\pi}{\sqrt{\lambda}} + \int_0^\pi \frac{\sin \sqrt{\lambda}(\pi - t)}{\sqrt{\lambda}} p(t)\omega(t, \lambda)\, dt.$$

The representation can be conveniently used for investigation of the asymptotic property of $\omega(\lambda)$ and of the eigenvalues λ_l, which are determined by the zeros of $\omega(\lambda)$.

It is not difficult to show that

$$\sum_{l=j+1}^{\infty} \lambda_l^s = \left[\frac{1}{\pi} p \int_{-\infty}^{0} \frac{\frac{d}{d\lambda}\omega(\lambda)}{\omega(\lambda)} \lambda^s d\lambda - i \sum_{l=1}^{j} \lambda_l^s\right] e^{-i\pi s} \sin \pi s;$$

where λ_l, $l = 1, \cdots, j$, are the negative eigenvalues; λ_l, $l = j + 1, \cdots$, are the positive eigenvalues; and $-1 < \operatorname{Re} s < -1/2$.

The analytic continuation of this formula, as in the case discussed above, leads to the identities for the eigenvalues.

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