On the Work of E. Witten

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It is a duty of the chairman of the Fields Medal Committee to appoint the speakers, who describe the work of the winners at this session. Professor M. Atiyah was asked by me to speak about Witten. He told me that he would not be able to come but was ready to prepare a written address. So it was decided that I shall make an exposition of his address adding my own comments. The full text of Atiyah's address is published separately.

Let me begin by the statement that Witten's award is in the field of Mathematical Physics.

Physics was always a source of stimulus and inspiration for Mathematics so that Mathematical Physics is a legitimate part of Mathematics. In classical time its connection with Pure Mathematics was mostly via Analysis, in particular through Partial Differential Equations. However quantum era gradually brought a new life. Now Algebra, Geometry and Topology, Complex Analysis and Algebraic Geometry enter naturally into Mathematical Physics and get new insights from it. And[1]

In all this large and exciting field, which involves many of the leading physicists and mathematicians in the world, Edward Witten stands out clearly as the most influential and dominating figure. Although he is definitely a physicist (as his list of publications clearly shows) his command of mathematics is rivalled by few mathematicians, and his ability to interpret physical ideas in mathematical form is quite unique. Time and again he has surprised the mathematical community by a brilliant application of physical insight leading to new and deep mathematical theorems.

Now I come to description of the main achievements of Witten. In Atiyah's text many references are given to Feynman Integral, so that I begin with a short and rather schematic reminding of this object.

In quantum physics the exact answers for dynamical problems can be expressed in a formal way as follows:

$$Z = \int e^{iA} \prod_x d\mu$$

where A is an action functional of local fields – functions of time and space variables x, running through some manifold M. The integration measure is a

[1] Small print type here and after refers to Atiyah's text.

product of local measures for values of fields in a point x over all M. The result of integration Z could be a number or function of parameters defining the problem – coupling constants, boundary or asymptotical conditions, etc.

In spite of being an ill-defined object from the point of view of rigorous mathematics, Feynman functional integral proved to be a powerful tool in quantum physics. It was gradually realized that it is also a very convenient mathematical means. Indeed the geometrical objects such as loops, connections, metrics are natural candidates for local fields and geometry produces for them interesting action functionals. The Feynman integral then leads to important geometrical or topological invariants.

Although this point of view was expressed and exemplified by several people (e.g., A. Schvarz used 1-forms ω on a three dimensional manifold with action $A = \int \omega d\omega$ to describe the Ray-Singer torsion) it was Witten who elaborated this idea to a full extent and showed the flexibility and universality of Feynman integral.

Now let me follow Atiyah in description of the main achievements of Witten in this direction.

1. Morse Theory

His paper [2] on supersymmetry and Morse theory is obligatory reading for geometers interested in understanding modern quantum field theory. It also contains a brilliant proof of the classic Morse inequalities, relating critical points to homology. The main point is that homology is defined via Hodge's harmonic forms and critical points enter via stationary phase approximation to quantum mechanics. Witten explains that "supersymmetric quantum mechanis" is just Hodge-de Rham theory. The real aim of the paper is however to prepare the ground for supersymmetric quantum field theory as the Hodge-de Rham theory of infinite-dimensional manifolds. It is a measure of Witten's mastery of the field that he has been able to make intelligent and skilful use of this difficult point of view in much of his subsequent work.

Even the purely classical part of this paper has been very influential and has led to new results in parallel fields, such as complex analysis and number theory.

2. Index Theorem

One of Witten's best known ideas is that the index theorem for the Dirac operator on compact manifolds should emerge by a formally exact functional integral on the loop space. This idea (very much in the spirit of his Morse theory paper) stimulated an extensive development by Alvarez-Gaumé, Getzler, Bismut and others which amply justified Witten's view-point.

3. Rigidity Theorems

Witten [7] produced an infinite sequence of such equations which arise naturally in the physics of string theories, for which the Feynman path integral provides a heuristic explanation of rigidity. As usual Witten's work, which was very precise and detailed in its formal aspects, stimulated great activity in this area, culminating in rigorous proofs of these new rigidity theorems by Bott and Taubes [1]. A noteworthy aspect of these proofs is that they involve elliptic function theory and deal with the infinite sequence of operators simultaneously rather than term by term. This is entirely natural from Witten's view-point, based on the Feynman integral.

4. Knots

Witten has shown that the Jones invariants of knots can be interpreted as Feynman integrals for a 3-dimensional gauge theory [11]. As Lagrangian, Witten uses the Chern-Simons function, which is well-known in this subject but had previously been used as an addition to the standard Yang-Mills Lagrangian. Witten's theory is a major breakthrough, since it is the only intrinsically 3-dimensional interpretation of the Jones invariants: all previous definitions employ a presentation of a knot by a plane diagram or by a braid.

Although the Feynman integral is at present only a heuristic tool it does lead, in this case, to a rigorous development from the Hamiltonian point of view. Moreover, Witten's approach immediately shows how to extend the Jones theory from knots in the 3-sphere to knots in arbitrary 3-manifolds. This generalization (which includes as a specially interesting case the empty knot) had previously eluded all other efforts, and Witten's formulas have now been taken as a basis for a rigorous algorithmic definition, on general 3-manifolds, by Reshetikin and Turaev.

Now I turn to another beautiful result of Witten – proof of positivity of energy in Einstein's Theory of Gravitation.

Hamiltonian approach to this theory proposed by Dirac in the beginning of the fifties and developed further by many people has led to a natural definition of energy. In this approach a metric γ and external curvature h on a space-like initial surface $S^{(3)}$ embedded in space-time $M^{(4)}$ are used as parameters in the corresponding phase space. These data are not independent. They satisfy Gauss-Codazzi constraints – highly nonlinear PDE. The energy H in the asymptotically flat case is given as an integral of indefinite quadratic form of $\nabla\gamma$ and h. Thus it is not manifestly positive. The important statement that it is nevertheless positive may be proved only by taking into account the constraints – a formidable problem solved by Yau and Schoen in the late seventies and as Atiyah mentions, "leading in part to Yau's Fields Medal at the Warsaw Congress".

Witten proposed an alternative expression for energy in terms of solution of a linear PDE with the coefficients expressed through γ and h. This equation is

$$\mathscr{D}^{(3)}\psi = 0$$

where $\mathscr{D}^{(3)}$ is the Dirac operator induced on $S^{(3)}$ by the full Dirac operator on $M^{(4)}$. Witten's formula somewhat schematically can be written as follows:

$$H(\psi_0, \psi_0) = \int (|\nabla\psi|^2 + \psi^* G\psi)dS$$

where ψ_0 is the asymptotic boundary value for ψ and G is proportional to the Einstein tensor $R_{ik} - \frac{1}{2}g_{ik}R$. Due to the equation of motion $G = T$, where T is the energy-momentum tensor of matter and thus manifestly positive. So the positivity of H follows.

This unexpected and simple proof shows another ability of Witten – to solve a concrete difficult problem by specific elegant means.