$$q$$

$$\sum_{n \geq 0} c_n q^n$$

$$\prod_k (1 - q^{k+1})^{-1} = \sum_n p(n) q^n$$

$$e_q(x) = \sum_n \frac{x^n}{(1-q)(1-q^2)\dots(1-q^n)}$$

$q$ – base for $q$- hypergeometric serieses

Heine, Jackson ... Askey, Rahman

$q$ in automorphic function

$$q = e^{i\pi\tau}$$

$(1, \tau)$ – periods

$\omega, \omega'$ $\tau = \frac{\omega'}{\omega}$, $\omega\omega' = -\frac{1}{4}$

New life $\quad q \quad$ quantum

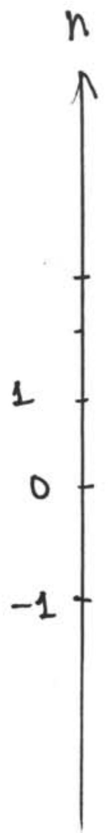
H. Weyl $\qquad uv = q^2 vu$

Quantum integrable models $\Rightarrow$

$\Rightarrow$ Quantum Groups

Plan:

1. Dynamical model
2. Solution in terms of $q$-exp
3. Modular double




$X_n$ real positive

$$X_{n+1} X_{n-1} = 1 + X_n^2$$

Cluster algebras ; CFT Liouville

$X_0, X_{-1}$ – initial data

$$\{X_0, X_{-1}\} = \gamma X_{-1} X_0$$

QM $\quad X_n$ - selfadjoint operators

$$X_{-1} X_0 = q X_0 X_{-1} \qquad q = e^{i\gamma\hbar}$$

$$\{\ ,\ \} = \frac{i}{\hbar} [\ ,\ ]$$

$$X_{n+1} X_{n-1} = 1 + q^{-1} X_n^2$$

$\mathfrak{h} = L_2(\mathbb{R}) \qquad f(z) \qquad$ Heisenberg $\qquad Q, P$

$$Q f(z) = z f(z) \qquad P f(z) = \frac{1}{2\pi i} \frac{df}{dz}$$

$$X_{-1} = \exp\{i\pi P/\omega\} \qquad X_0 = \exp\{-i\pi Q/2\omega\}$$

$$X_{-1} f(z) = f(z - 2\omega') \qquad X_0 f(z) = e^{-i\pi z/2\omega} f(z)$$

Unbounded for imaginary $\omega, \omega'$

Evolution $\qquad X_n = U^{-n} X_0 U^n$

$$U = e^{-iH}$$ one step is enough

$$X_{-1} = x \,, \qquad X_0 = y$$

$$U^{-1} x U = y \,, \qquad U^{-1} y U = (1 + q^{-1} y^2) x^{-1}$$

$$U = V\, W$$

$$x V = V y \qquad y V = V x^{-1}$$

Fourier

$$[W, y] = 0 \qquad W = E(y^2)$$

$$x U = U y\,, \qquad y U = y V W = V x^{-1} E(y^2) =$$

$$= V \underbrace{E(q^{-2} y^2)} x^{-1} = V \underbrace{E(y^2)(1 + q^{-1} y^2)} x^{-1}$$

$$\frac{E(q^{-2} w)}{E(w)} = 1 + q^{-1} w \qquad (w = y^2)$$

$$\frac{E(qw)}{E(q^{-1}w)} = \frac{1}{1+w}$$

$$E_q(w) = \prod_{k\geq 0} \left(1 + q^{2k+1} w\right)$$

Schützenberger $\quad uv = q^2 vu$

$$E(u)\,E(v) = E(u+v)$$

More recent (FV, GF)

$$E(v)\,E(u) = E(u+v+q^{-1}uv) = E(u)\,E(q^{-1}uv)\,E(v$$

# Noncommutative 3 - cocycle

$$E(w) = \exp\left\{ \frac{1}{4} \sum_{k \geq 0} \frac{(-1)^k w^k}{k(q^k - q^{-k})} \right\}$$

q - Dilogarithm ! FK

Grave complication $|q| < 1$

Way out : $\Gamma(w) = \dfrac{E_q(w)}{E_{\tilde{q}}(\tilde{w})}$

$\tilde{q} = e^{-i\pi/\tau}$ $\tilde{w} = w^{1/\tau}$

$$\Gamma\left(e^{-i\pi z/\omega}\right) = \gamma(z)$$

$$\gamma(z) = \exp\left\{ \frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{-2izt}}{\sin \omega t \, \sin \omega' t} \frac{dt}{t} \right\}$$

All nonreal $\omega, \omega'$

Symmetric $\omega \Leftrightarrow \omega'$ or $\tau \Leftrightarrow 1/\tau$

$y^{2/\tau}$ commutes with $x$ and $y$.

Second dynamical system $\tilde{X}_n = X_n^{1/\tau}$

$\mathcal{A}_q$ generated by $x, y$

$\mathcal{A}_{\tilde{q}}$ generated by $\tilde{x}, \tilde{y}$

$$\mathcal{B} = \mathcal{A}_q \otimes \mathcal{A}_{\tilde{q}}$$

$\omega, \omega'$ – imaginary $\quad |\gamma(z)| = 1.$ Regim I

$\omega = -\overline{\omega'}$ $\quad$ — " — $\quad$ Regim II

$|\tau| = 1$; $\quad x^* = \tilde{x}, \; y^* = \tilde{y}$

Region I $\quad (a \otimes b)^* = a^* \otimes b^*$

Region II $\quad (a \otimes b)^* = b^* \otimes a^*$

Modular duality

Liouville model

In CFT

$$c = 1 + 6\left(\tau + \frac{1}{\tau} + 2\right)$$

$\tau > 0 \quad c > 25$

$\tau < 0 \quad c < 1$

$|\tau| = 1 \quad 1 < c < 25$

Polyakov noncritical string

Representations of $U_q(SL(2,\mathbb{R})) \otimes U_{\tilde{q}}(SL(2\mathbb{R}))$ 

Penthagon : $\gamma(Q)\gamma(P) = \gamma(P)\gamma(P+Q)\gamma(Q)$

Yang - Baxter realization

Quantum Teichmüller K, F-ch .

$$(u + u^{-1} + v)\psi = \lambda\psi$$

$q$ - Plancherel

# Barnes double-sine

$$\frac{\gamma(z+\omega')}{\gamma(z-\omega')} = 1 + e^{-i\pi z/\omega}$$

$$\frac{\gamma(z+\omega)}{\gamma(z-\omega)} = 1 + e^{-i\pi z/\omega'}$$

$$\gamma(z)\gamma(-z) = e^{-i\pi z^2}$$

$$\gamma(z+\omega'') = \int_{-\infty}^{\infty} dt\, e^{-2\pi i z t} \frac{1}{\gamma(\omega''-t)}$$

Continual analogues of addition theorems

A. Volkov „Noncommutative hypergeometry"