STRONGLY COUPLED QUANTUM DISCRETE LIOUVILLE THEORY. II: GEOMETRIC INTERPRETATION OF THE EVOLUTION OPERATOR

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Abstract. It is shown that the N-th power of the light-cone evolution operator of $2N$-periodic quantum discrete Liouville model can be identified with the Dehn twist operator in quantum Teichmüller theory.

Introduction

Integrable lattice regularization of quantum Liouville theory has been developed in papers [4, 2, 5]. According to recent development in [3], the model is expected to describe quantum Liouville equation with Virasoro central charge $c_L > 1$, including the "strongly coupled regime" $1 < c_L < 25$.

This paper can be considered as a second part of our previous work [3] dedicated to discrete Liouville model. Here we show that the evolution operator of the model can be interpreted in pure geometrical terms within quantum Teichmüller theory [6, 1, 7, 8, 9]. Namely, we identify the N-th power of the light-cone evolution operator of quantum discrete Liouville model of spatial length $2N$ (which is the number of sites in a chain) with the Dehn twist operator in quantum Teichmüller theory.

The paper is organized as follows. The quantum discrete Liouville system is briefly described in Section 1. The relation to quantum Teichmüller theory is explained in Section 2.

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1. Quantum discrete Liouville system

1.1. **Algebra of observables.** Following [3], algebra of observables $\mathcal{A}_N$, $N > 1$, is generated by selfadjoint elements f_j, $j \in \mathbb{Z}$, with periodicity condition $\mathsf{f}_{j+2N} = \mathsf{f}_j$ and commutation relations

$$[\mathsf{f}_m, \mathsf{f}_n] = \begin{cases} (-1)^m (2\pi \mathsf{i})^{-1}, & \text{if } n = m \pm 1 \pmod{2N} \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

1.2. **Equations of motion.** The field variables

$$\chi_{j,t} \equiv \mathsf{U}_{lc}^t e^{2\pi b \mathsf{f}_{j+t}} \mathsf{U}_{lc}^{-t}, \quad j + t = 1 \pmod 2$$

are defined so that

$$\mathsf{U}_{lc} \chi_{j,t} \mathsf{U}_{lc}^{-1} = \chi_{j-1,t+1}$$

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Here the "light-cone" evolution operator U_{lc} is defined explicitly

$$\mathsf{U}_{lc} = \mathsf{G}\prod_{j=1}^{N}\varphi_b(\mathsf{f}_{2j}) = \prod_{j=1}^{N}\varphi_b(\mathsf{f}_{2j-1})\mathsf{G} \tag{1.2}$$

where

$$\varphi_b(z) \equiv \exp\left(\frac{1}{4}\int_{\mathsf{i}0-\infty}^{\mathsf{i}0+\infty}\frac{e^{-\mathsf{i}2zw}\,dw}{\sinh(wb)\sinh(w/b)w}\right)$$
$$= (e^{2\pi(z+c_b)b};q^2)_\infty/(e^{2\pi(z-c_b)b^{-1}};\bar{q}^2)_\infty,$$

$$q \equiv e^{\mathsf{i}\pi b^2},\quad \bar{q} \equiv e^{-\mathsf{i}\pi b^{-2}},\quad c_b \equiv \mathsf{i}(b+b^{-1})/2$$

while operator G is defined through the equations

$$\mathsf{G}\mathsf{f}_j = (-1)^j\mathsf{f}_{j-1}\mathsf{G} \tag{1.3}$$

The field variables solve the quantum discrete Liouville equation [1]

$$\chi_{j,t+1}\chi_{j,t-1} = (1+q\chi_{j+1,t})(1+q\chi_{j-1,t})$$

with spatial periodic boundary condition

$$\chi_{j+2N,t} = \chi_{j,t}$$

and the initial data given by exponentiated generators

$$\chi_{2j+1,0} = e^{2\pi b\mathsf{f}_{2j+1}},\quad \chi_{2j,-1} = e^{2\pi b\mathsf{f}_{2j}}$$

2. Interpretation within quantum Teichmüller theory

In this section we interpret the evolution operator U_{lc} in geometrical terms by using the formalism of decorated ideal triangulations and their transformations within quantum Teichmüller theory described in [9].

2.1. **Geometric realization.** We consider an annulus with N marked points on each of its boundary components ($2N$ points in total) and choose decorated ideal triangulation τ_N of it shown in Fig. 1. Equivalently, we can think of τ_N as an

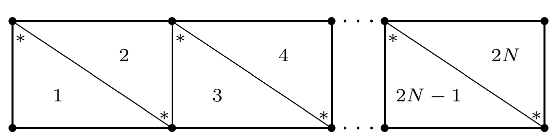


Figure 1. Decorated ideal triangulation τ_N of an annulus with N marked points on each boundary component. The leftmost and the rightmost vertical edges are identified.

infinite triangulated strip where triangles are numerated by integers in accordance with Fig. 1 with periodicity condition $\bar{\tau}_N(n+2N) = \bar{\tau}_N(n)$, $\forall n \in \mathbb{Z}$. In this way we come to identification of the integers from 1 to $2N$, numbering triangles in τ_N, with elements of the ring of residues $\mathbb{Z}_{2N} \equiv \mathbb{Z}/2N\mathbb{Z}$. Such identification will be assumed in algebraic expressions, when necessary.

Denote $D^{1/N}$ the isotopy class of a homeomorphism of the annulus which rotates the top boundary component wrt the bottom thru angle $2\pi/N$ so that the marked

[1] Using invariance of U_{lc} with respect to symmetry $b \leftrightarrow b^{-1}$, one can also define the dual fields satisfying the dual equation, see [3].

points of the top boundary are cyclically shifted by one period. The reason for using fractional power notation comes from the fact that

$$\underbrace{D^{1/N} \circ \cdots \circ D^{1/N}}_{N \text{ times}} = D$$

is nothing else but the Dehn twist. From Fig. 2 it follows that the following com-

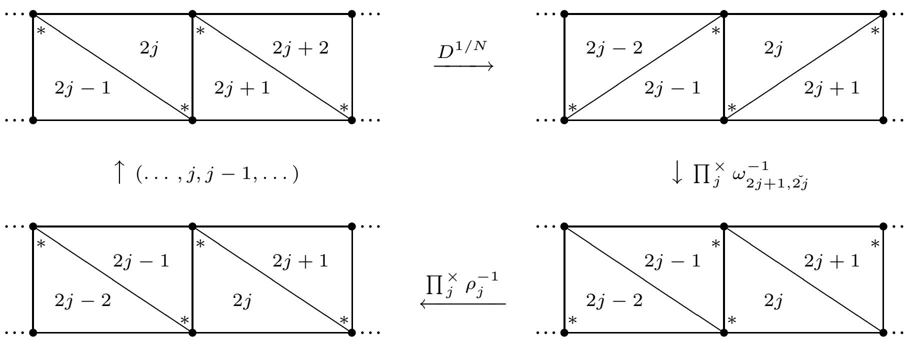
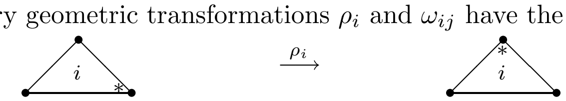


FIGURE 2. Continuous transformation $D^{1/N}$ of triangulated annulus τ_N is a cyclic shift of the top boundary wrt the bottom boundary to the right by one spacing.

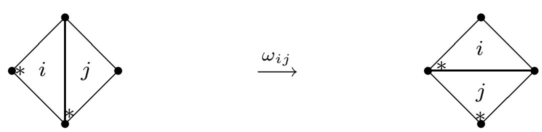
position of geometric transformations is identity,

$$(\dots, j, j-1, \dots) \circ \prod_k^{\times} \rho_k^{-1} \circ \prod_l^{\times} \omega_{2l+1,\check{2l}}^{-1} \circ D^{1/N} = id$$

where elementary geometric transformations ρ_i and ω_{ij} have the form



and



with

$$\omega_{k,\check{l}} \equiv \rho_l \circ \omega_{k,l} \circ \rho_l^{-1}$$

while $(\dots, j, j-1, \dots)$ denotes the index shift transformation $j \mapsto j-1$. Equivalently we can write

$$D^{1/N} = \prod_l^{\times} \omega_{2l+1,\check{2l}} \circ \prod_k^{\times} \rho_k \circ (\dots, j, j+1, \dots)$$

so, quantum realization of $D^{1/N}$ in $L^2(\mathbb{R}^{2N})$ has the form

$$\mathsf{F}\left(\tau_N, D^{1/N}(\tau_N)\right) \simeq \mathsf{D}^{1/N} \equiv \zeta^{-N-6/N} \mathsf{P}_{(\dots j,j+1\dots)} \prod_{k=1}^{2N} \mathsf{A}_k \prod_{l=1}^{N} \mathsf{T}_{2l+1,\check{2l}} \tag{2.1}$$

with the normalization factor chosen in accordance with the convention for Dehn twists used in [8]. Here $\mathsf{P}_{(\dots j,j+1\dots)}$ is the natural realization of the cyclic permutation,

$$\zeta = e^{-\mathrm{i}\pi(b+b^{-1})^2/12}$$

$$\mathsf{A}_k \equiv e^{-\mathrm{i}\pi/3} e^{\mathrm{i}3\pi\mathsf{q}_k^2} e^{\mathrm{i}\pi(\mathsf{p}_k+\mathsf{q}_k)^2}$$

$$\mathsf{T}_{k,\check{l}} = e^{-\mathrm{i}2\pi\mathsf{p}_k\mathsf{p}_l}\, \bar{\varphi}_b(\mathsf{q}_k + \mathsf{q}_l), \quad \bar{\varphi}_b(z) \equiv (\varphi_b(z))^{-1}$$

where selfadjoint operators $\mathsf{p}_j, \mathsf{q}_j$ satisfy Heisenberg commutation relations

$$[\mathsf{p}_j, \mathsf{p}_k] = [\mathsf{q}_j, \mathsf{q}_k] = 0, \quad [\mathsf{p}_j, \mathsf{q}_k] = \delta_{j,k}(2\pi \mathsf{i})^{-1}$$

We can rewrite eqn (2.1) in the form

$$\text{(2.2)} \quad \mathsf{D}^{-1/N} = \zeta^{N+6/N} \prod_{m=1}^{N} \varphi_b(\mathsf{q}_{2m} + \mathsf{q}_{2m+1}) e^{\mathsf{i}2\pi \sum_{j=1}^{N} \mathsf{p}_{2j}\mathsf{p}_{2j+1}} \prod_{k=1}^{2N} \mathsf{A}_k^{-1} \mathsf{P}_{(\dots,l,l-1,\dots)}$$

Proposition 1. *Operators*

$$\text{(2.3)} \qquad \kappa(\mathsf{f}_j) = \begin{cases} \mathsf{p}_j + \mathsf{p}_{j-1}, & \text{if } j = 0 \pmod 2 \\ \mathsf{q}_j + \mathsf{q}_{j-1}, & \text{otherwise} \end{cases}$$

$$\text{(2.4)} \qquad \kappa(\mathsf{G}) = \zeta^{N+6/N} e^{\mathsf{i}2\pi \sum_{j=1}^{N} \mathsf{p}_{2j}\mathsf{p}_{2j+1}} \prod_{k=1}^{2N} \mathsf{A}_k^{-1} \mathsf{P}_{(\dots,l,l-1,\dots)}$$

define faithful (reducible) realization of the observable algebra $\mathcal{A}_N$ *in* $L^2(\mathbb{R}^{2N})$.

Proof. Let us check that these definitions are consistent with relations (1.1), (1.3). First, evidently,

$$[\kappa(\mathsf{f}_{2j}), \kappa(\mathsf{f}_{2k})] = [\kappa(\mathsf{f}_{2j+1}), \kappa(\mathsf{f}_{2k+1})] = 0$$

while

$$[\kappa(\mathsf{f}_{2j}), \kappa(\mathsf{f}_{2k+1})] = [\mathsf{p}_{2j} + \mathsf{p}_{2j-1}, \mathsf{q}_{2k+1} + \mathsf{q}_{2k}] = (2\pi\mathsf{i})^{-1}(\delta_{j,k} + \delta_{j,k+1})$$

thus reproducing relations (1.1). Next,

$$\mathrm{Ad}(\kappa(\mathsf{G}))\kappa(\mathsf{f}_{2j}) = \mathrm{Ad}(\kappa(\mathsf{G}))(\mathsf{p}_{2j} + \mathsf{p}_{2j-1}) = \mathrm{Ad}(e^{\mathsf{i}2\pi \sum_{k=1}^{N} \mathsf{p}_{2k}\mathsf{p}_{2k+1}})(\mathsf{p}_{2j-1} + \mathsf{p}_{2j-2})$$
$$= \mathrm{Ad}(e^{\mathsf{i}2\pi \sum_{k=1}^{N} \mathsf{p}_{2k}\mathsf{p}_{2k+1}})(\mathsf{q}_{2j-1} - \mathsf{p}_{2j-1} + \mathsf{q}_{2j-2} - \mathsf{p}_{2j-2}) = \mathsf{q}_{2j-1} + \mathsf{q}_{2j-2} = \kappa(\mathsf{f}_{2j-1})$$

and similarly

$$\mathrm{Ad}(\kappa(\mathsf{G}))\kappa(\mathsf{f}_{2j+1}) = \mathrm{Ad}(\kappa(\mathsf{G}))(\mathsf{q}_{2j+1} + \mathsf{q}_{2j})$$
$$= \mathrm{Ad}(e^{\mathsf{i}2\pi \sum_{k=1}^{N} \mathsf{p}_{2k}\mathsf{p}_{2k+1}})(\mathsf{q}_{2j} + \mathsf{q}_{2j-1}) = -\mathsf{p}_{2j} - \mathsf{p}_{2j-1} = -\kappa(\mathsf{f}_{2j})$$

in agreement with eqn (1.3). □

Now, comparing eqns (1.2), (2.2), we come to our main result

$$\text{(2.5)} \qquad \kappa(\mathsf{U}_{lc}) = \mathsf{D}^{-1/N}$$

2.2. **A similarity transformation.** Here, we give (without proof) the result of similarity transformation which simplifies the N-th power of the evolution operator.

Define

$$\mathsf{W} \equiv \prod_{N \geq j > 1} (\bar{\varphi}_b(\mathsf{f}_{2j})\, \varphi_b(\mathsf{f}_{2j-1})\, \varphi_b(\mathsf{g}_{2j,2N+1})), \quad \bar{\varphi}_b(x) \equiv (\varphi_b(x))^{-1}$$

where $\mathsf{g}_{j,k} \equiv \sum_{l=j+1}^{k} \mathsf{f}_j$, and the product of noncommuting operators is in decreasing order from left to right.

Proposition 2. *One has the following explicit expression for the transformed evolution operator*

$$\tilde{\mathsf{U}}_{lc} \equiv \mathrm{Ad}(\mathsf{W}^{-1})\mathsf{U}_{lc} = (\zeta e^{\mathsf{i}\pi/6})^{-N} \prod_{N>k>1} \bar{\varphi}_b(\mathsf{g}_{2k-1,2N-1})$$
$$\times\, \bar{\varphi}_b(\mathsf{g}_{2,2N-1})\, \varphi_b(\mathsf{g}_{2N-1,2N+1})\, \bar{\varphi}_b(\mathsf{f}_2) e^{\mathsf{i}\pi \sum_{l=1}^{N} \mathsf{f}_{2l}^2} \mathsf{G}$$

where the product is again in decreasing order from left to right, while the N-th power has the form

$$\tilde{\mathsf{U}}_{lc}^N = (\zeta e^{\mathrm{i}\pi/6})^{1-N^2}\, \varphi_b(\mathsf{g}_{2,2N+1}) e^{-\mathrm{i}\pi \mathsf{f}_2^2} \left(e^{\mathrm{i}\pi \sum_{l=1}^N \mathsf{f}_{2l}^2} \mathsf{G} \right)^N$$

Conclusion

The main result of this paper is formula (2.5) which, on one hand side, identifies "zero-modes" of $2N$-periodic quantum discrete Liouville equation to be given by N-th power of the light-cone evolution operator U_{lc}, and equates these zero-modes to the (inverse of) Dehn twist operator in quantum Teichmüller theory, on the other. Immediate consequence of this result is that now, based on the known spectrum of operator D, we know the spectrum of the model. Indeed, the complete spectrum of D is given by the formula [9]:

$$\mathrm{Spec}(\mathsf{D}) = \{e^{\mathrm{i}2\pi\Delta_s} | s \in \mathbb{R}_{>0}\}$$

where

$$\Delta_s = \frac{c_L - 1}{24} + s^2, \quad c_L = 1 + 6(b + b^{-1})^2 > 1$$

are conformal weights and the Virasoro central charge in (continuous) quantum Liouville theory, see [10] for a recent review. This is consistent with interpretation of the Dehn twist spectrum as Liouville conformal weights through the formula $\mathrm{Spec}(\mathsf{D}) = \mathrm{Spec}(e^{\mathrm{i}2\pi L_0})$, where L_0 is the Virasoro generator in continuous quantum Liouville theory with the known spectrum

$$\mathrm{Spec}(L_0) = \{\Delta_s + m | s \in \mathbb{R}_{>0},\ m \in \mathbb{Z}_{\geq 0}\}$$

Our result implies the following spectrum of U_{lc}:

$$\mathrm{Spec}(\mathsf{U}_{lc}) = \{e^{-\mathrm{i}2\pi(\Delta_s+m)/N} | s \in \mathbb{R}_{>0},\ m \in \mathbb{Z}/N\mathbb{Z}\}$$

which coincides with the spectrum of the exponential operator $e^{-2\pi\mathrm{i}L_0/N}$ in quantum Liouville theory. Thus, the discrete version of quantum Liouville theory is in complete agreement with the continuous one and there is no any modification in the spectrum of conformal weights.

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