

SHAFRANOV'S VIRIAL THEOREM AND MAGNETIC PLASMA CONFINEMENT

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Shafranov's virial theorem implies that nontrivial magnetohydrodynamical equilibrium configurations must be supported by externally supplied currents. Here we extend the virial theorem to field theory, where it relates to Derrick's scaling argument on soliton stability. We then employ virial arguments to investigate a realistic field theory model of a two-component plasma, and conclude that stable localized solitons can exist in the bulk of a finite density plasma. These solitons entail a nontrivial electric field which implies that purely magnetohydrodynamical arguments are insufficient for describing stable, nontrivial structures within the bulk of a plasma.

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Ideal single fluid magnetohydrodynamics obeys an integral relation which is known as Shafranov's virial theorem [1], [2]. It implies that an ideal magnetohydrodynamical system can not support nontrivial localized structures. Instead any nontrivial equilibrium configuration must be maintained by externally supplied currents, a guiding principle in the design of contemporary magnetic fusion devices.

Ideal magnetohydrodynamics is supposedly adequate for describing the ground state equilibrium geometry of a plasma. It also provides a starting point for a weak coupling Boltzmannian transport theory [2]. But as an effective mean field theory it lacks the kind of detailed microscopic information which is needed to properly account for the electromagnetic interactions between the charged particles within the plasma. For this, ideal magnetohydrodynamics should be replaced by an appropriate classical field theory of charged particles. With a firm microscopic basis and established set of rules for systematic computations, a field theory model can provide a rigorous basis for describing thermal fluctuations and dynamical effects, including transport phenomena and issues related to plasma stability and confinement.

Here we extend Shafranov's virial theorem to classical field theory where it yields a variant of Derrick's scaling argument, widely employed to inspect soliton stability. We then apply virial arguments to a realistic field theory model of plasma. In accordance with ideal magnetohydrodynamics, we conclude that the field theory does not support localized self-confined plasma configurations in isolation, in an otherwise empty space. But when we inspect the finite density bulk properties of the field theoretical plasma, we find that the virial theorem does allow for the existence of stable solitons. These solitons describe extended collective excitations of charged particles in the otherwise uniform finite density environment. Our results are consistent with a recent proposal [3], that a finite density field theoretical plasma supports stable knotted solitons [4]. Indeed, we expect that these solitons can be employed to describe a variety of observed phenomena. For example coronal loops that are present in solar photosphere are natural candidates. The properties of these solitons may also become attractive in fusion experiments, where their stability might help in the design of particularly stable magnetic geometries [2].

Shafranov's virial theorem [1] follows from the properties of the magnetohydrodynamical energy-momentum tensor $T^{\mu\nu}$ in the ideal single fluid approximation. Its spatial components are [2]

$$T^{ij} = \rho v^i v^j + \left(p + \frac{1}{2}B^2\right) \delta^{ij} - B^i B^j \quad (1)$$

while the purely temporal component coincides with the internal energy density,

$$T^{00} = \frac{1}{2}\rho v^2 + \frac{1}{2}B^2 + \frac{p}{\gamma - 1} \quad (2)$$

Here γ is the ratio of specific heats. The fluid variables are the mass density ρ , the (bulk) fluid velocity v^i and the pressure p , and B^i is the magnetic field in natural

units with $\mu_0 = 1$. The plasma evolves according to the Navier-Stokes equation which follows when we equate the divergence of the energy-momentum tensor with external dissipative forces. These dissipative forces are present whenever the plasma is in motion, but cease when the plasma reaches a stable magnetostatic equilibrium configuration that minimizes the internal energy

$$E = \int d^3x T^{00} \quad (3)$$

Shafranov's virial theorem follows when we subject (3) to a scale transformation of the spatial coordinates $x^i \rightarrow \lambda x^i$ with λ a constant. For the magnetic field we select $B_i(x) \rightarrow \lambda^2 B_i(\lambda x)$, as customary in Maxwell's theory. But for the pressure p ideal magnetohydrodynamics does not supply enough information to determine its behaviour under a scale transformation. For this we assume that the pressure is subject to the standard thermodynamic scaling relation of a thermally isolated gas,

$$pV^\gamma = \text{constant} \quad (4)$$

This implies that under a scaling $p(x) \rightarrow \lambda^{3\gamma} p(\lambda x)$. If we assume that the value $\lambda = 1$ corresponds to an actual minimum energy configuration of the energy (3), when viewed as a function of λ the energy (3) then has an extremum at $\lambda = 1$. Consequently

$$0 = \left. \frac{\delta E(\lambda)}{\delta \lambda} \right|_{\lambda=1} = - \int d^3x \left(3p + \frac{1}{2} B^2 \right) \equiv - \int d^3x T^i_i \quad (5)$$

The magnetic contribution to the pressure is manifestly positive definite. Furthermore, (collisionless) kinetic theory relates the pressure p to the kinetic energy of the individual particles, which is similarly a positive definite quantity. The integrand in (5) is then positive definite, and we conclude that under the present assumptions non-trivial localized equilibrium configurations do not exist in ideal magnetohydrodynamics [1], [2].

We have formulated our derivation of Shafranov's virial theorem so that it relates to Derrick's scaling argument in classical field theory [5]. For this, we consider a generic three-dimensional Hamiltonian field theory model with classical action

$$S = \int dt d^3x \mathcal{L}(\psi) = \int dt d^3x \{ \pi_\alpha \dot{\varphi}^\alpha - H[\pi, \varphi] \} \quad (6)$$

The fields $\psi_\alpha \sim (\pi_\alpha, \varphi^\alpha)$ are canonical conjugates with Poisson bracket

$$\{ \pi_\alpha(x), \varphi^\beta(y) \} = \delta_\alpha^\beta \delta(x - y) \quad (7)$$

Notice that the time derivative in (6) acts asymmetrically. But in the sequel it will be useful to consider symmetrized quantities, and for this we generalize the time derivative term by a canonical transformation into

$$\int dt d^3x \pi_\alpha \dot{\varphi}^\alpha \rightarrow \int dt d^3x [a \dot{\pi}_\alpha \varphi^\alpha + (1 - a) \pi_\alpha \dot{\varphi}^\alpha] \quad (8)$$

where a parametrizes the canonical transformation.

We assume that the Hamiltonian H is a functional of the fields $\psi = (\pi, \varphi)$ and their first derivatives only, with no explicit dependence on the space coordinates x^i and time $t \equiv x^0$. We then obtain the energy-momentum tensor directly from Noether's theorem: Since there is no explicit dependence on x^μ

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\delta \mathcal{L}}{\delta \psi_\alpha} \partial_\mu \psi_\alpha + \frac{\delta \mathcal{L}}{\delta \partial_\nu \psi_\alpha} \partial_\mu \partial_\nu \psi_\alpha \quad (9)$$

and by employing the equations of motion we identify the components of the energy-momentum tensor with the ensuing four conserved currents

$$T^\mu{}_\nu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi_\alpha} \partial_\nu \psi_\alpha - \delta^\mu{}_\nu \mathcal{L} \quad (10)$$

In general (10) fails to be symmetric. But in the following we find it useful to consider symmetrized quantities, and for this we can re-define

$$T^\mu{}_\nu \rightarrow T^\mu{}_\nu + \partial_\rho X^{\rho\mu}{}_\nu \quad (11)$$

where $X^{\rho\mu}{}_\nu = -X^{\mu\rho}{}_\nu$ has no effect on the dynamics. (Note that if the theory fails to be Lorentz invariant, in general there will be no symmetry between the momentum flux $T^0{}_i$ and the energy flux $T^i{}_0$.)

We are interested in a scale transformation of the spatial coordinates $x^i \rightarrow \lambda x^i$, which sends $\psi_\alpha(x) \rightarrow \lambda^{D_\alpha} \psi_\alpha(\lambda x)$. Here D_α is the scale dimension of the field ψ_α . By considering an infinitesimal transformation with $\lambda = 1 + \epsilon$ we find for the generator δ_S of the scale transformation

$$\delta_S \pi_\alpha = x^i \partial_i \pi_\alpha + D_\pi^\alpha \pi_\alpha \quad (12)$$

$$\delta_S \varphi^\alpha = x^i \partial_i \varphi^\alpha + D_\alpha^\varphi \varphi^\alpha \quad (13)$$

For the energy density this yields

$$\delta_S T^0{}_0 = \left\{ -T^i{}_i + \partial_i (x^i T^0{}_0) + \sum_\alpha D_\alpha \left[\frac{\delta T^0{}_0}{\delta \psi_\alpha} \psi_\alpha + \frac{\delta T^0{}_0}{\delta \partial_k \psi_\alpha} \partial_k \psi_\alpha \right] \right\} \quad (14)$$

In general the scale dimensions can be arbitrary, and there is no *a priori* relation between the different D_α . But if the scale transformation is a canonical transformation it must preserve Poisson brackets, which implies that the scale dimensions of a canonical pair are subject to

$$D_\pi^\alpha + D_\alpha^\varphi = 3 \quad (15)$$

The generator δ_S^C of such a canonical scale transformation can be computed from Noether's theorem, and by properly selecting the value of a in (8) we arrive at the symmetrized form

$$\delta_S^C = \int d^3x x^i T^0{}_i \quad (16)$$

We are interested in a refinement of ideal magnetohydrodynamics, a microscopic field theory model of a two-component plasma with negatively charged electrons (e) and positively charged ions (i) and classical (first-order) Lagrangian [3],

$$\begin{aligned} \mathcal{L} = & E_k \partial_t A_k + \frac{i}{2} (\psi_e^* \partial_t \psi_e - \partial_t \psi_e^* \psi_e + \psi_i^* \partial_t \psi_i - \partial_t \psi_i^* \psi_i) - \frac{1}{2} E_k^2 - \frac{1}{2} B_k^2 \\ & - \frac{1}{2m} |(\partial_k + ieA_k)\psi_e|^2 - \frac{1}{2M} |(\partial_k - ieA_k)\psi_i|^2 + A_0 (\partial_k E_k - e\psi_e^* \psi_e + e\psi_i^* \psi_i) \end{aligned} \quad (17)$$

Here ψ_e and ψ_i are (complex) non-relativistic Hartree-type fields that describe electrons and ions with masses m and M and electric charges $\pm e$ respectively, together with their electromagnetic interactions. Note that we have realized Maxwell's theory canonically so that the electric field E_i and spatial gauge field A_i form a canonical pair, with the temporal A_0 a Lagrange multiplier that enforces Gauss' law. Since the time derivative appears linearly in the charged fields, the action (17) admits a proper Hamiltonian interpretation with $\psi_{e,i}^*$ the canonical conjugates of $\psi_{e,i}$. Notice that for definiteness we have chosen both charged fields to be commuting. This should be adequate in the Boltzmannian limit, relevant in conventional plasma scenarios where the temperature is sufficiently high so that bound states (hydrogen atom) are prevented but not high enough for relativistic corrections to become important. Notice that we have also introduced an appropriate symmetrization of the form (8) in the time derivative terms of the charged fields. Finally, besides the terms that we have displayed in (17) we implicitly assume the presence of chemical potential terms that ensure overall charge neutrality. However, for the present purposes such terms are redundant and will either remain implicit, or will be enforced by appropriate boundary conditions.

We propose that the advantage of (17) over ideal magnetohydrodynamics is, that (17) provides a firm microscopic basis for systematically computing various properties of a plasma. For example an appropriate version of the equation of state (4) can be derived from (17). In particular, (17) yields immediately the standard electromagnetic many-body Schrödinger equation for a gas of electrons and ions.

The energy-momentum tensor T^μ_ν can be computed directly from (10). After we introduce an appropriate symmetrization which ensures manifest gauge invariance, we find for the energy density

$$T^0_0 = \frac{1}{2\mu} \{ \sin^2 \alpha |D_k \psi_e|^2 + \cos^2 \alpha |D_k^* \psi_i|^2 \} + \frac{E^2}{2} + \frac{B^2}{2} - A_0 (\partial_i E_i + e[\psi_i^* \psi_i - \psi_e^* \psi_e]) \quad (18)$$

where $D_k = \partial_k + ieA_k$ and $\mu = m \sin^2 \alpha = M \cos^2 \alpha$ is the reduced mass. For the spatial components of the energy-momentum tensor we find similarly, with the help of the equations of motion

$$T^i_k = E_i E_k + B_i B_k - \frac{1}{2\mu} \left\{ \sin^2 \alpha [(D_i \psi_e)^* (D_k \psi_e) + (D_k \psi_e)^* (D_i \psi_e)] \right.$$

$$+ \cos^2 \alpha [(D_i^* \psi_i)^* (D_k^* \psi_i) + (D_k^* \psi_i)^* (D_k^* \psi_i)] \} - \delta_k^i \mathcal{L} \quad (19)$$

Finally, for the generator of the canonical scale transformation we get

$$\delta_S^C = \int d^3x x^k T^0_k = \int d^3x x^k \left[E_i F_{ki} + \frac{i}{2} \{ \psi_e^* D_k \psi_e - D_k^* \psi_e^* \psi_e + \psi_i^* D_k^* \psi_i - D_k \psi_i^* \psi_i \} \right] \quad (20)$$

It yields the following gauge covariantized version of (12), (13),

$$\delta_S^C E_k = x^i \partial_i E_k + 2E_k + x^k (\partial_i E_i + e[\psi_i^* \psi_i - \psi_e^* \psi_e]) \quad (21)$$

$$\delta_S^C A_k = x^i \partial_i A_k + A_k - \partial_k (x^i A_i) \quad (22)$$

$$\delta_S^C \psi_{e,i} = x^i \partial_i \psi_{e,i} + \frac{3}{2} \psi_{e,i} \pm i e x^i A_i \psi_{e,i} x^i \partial_i \psi_{e,i}^* + \frac{3}{2} \psi_{e,i}^* \quad (23)$$

In particular, for each of the canonical variable $(\psi_{e,i}, \psi_{e,i}^*)$ the scale dimension is 3/2 so that the canonical scale generator commutes with the number operators for the charged particles

$$\{\delta_S^C, N_{e,i}\} = \delta_S^C \int d^3x \psi_{e,i}^* \psi_{e,i} = 0 \quad (24)$$

We now proceed to inspect the consequences of Shafranov's virial arguments. For this we remind that a static minimum energy configuration must be a stationary point of the energy (3), (18) under any *local* variation of the fields. Since the scale transformation (12), (13) is a non-local variation it does not need to leave the energy intact, unless it also preserves the pertinent boundary conditions. To determine these boundary conditions, we consider the plasma in two different physical environments:

In the first scenario we have an isolated, localized plasma configuration in an otherwise empty space, with a definite number of charged particles

$$N_e + N_i = \int d^3x (\psi_e^* \psi_e + \psi_i^* \psi_i) \quad (25)$$

Since the canonical scale generator commutes with the individual number operators (24), the ensuing variation of the fields is consistent with the boundary condition that the number of particles remains intact. By a direct computation we then find for a static stationary point of the energy,

$$\begin{aligned} 0 &= \delta_S^C \int d^3x T^0_0 = - \int d^3x T^i_i \\ &= - \int d^3x \left(-\frac{1}{\mu} \{ \sin^2 \alpha |D_k \psi_e|^2 + \cos^2 \alpha |D_k^* \psi_i|^2 \} - \frac{E^2}{2} - \frac{B^2}{2} \right) \end{aligned} \quad (26)$$

Since the trace of the spatial stress tensor is a sum of positive definite terms, in analogy with Shafranov's virial theorem in ideal magnetohydrodynamics (5) we conclude that there can not be any nontrivial stationary points. This means that in an otherwise

empty space an initially localized plasma configuration can not be confined solely by its internal electromagnetic interactions. additional interactions such as gravity must be present. Otherwise the canonical scale transformation dilutes the plasma by expanding its volume while keeping the number of the charged particles intact, until the collective behaviour of the plasma becomes replaced by an individual-particle behaviour of the charged constituents.

The second physical scenario of interest to us describes the bulk properties of a plasma: We are interested in an initially localized plasma configuration, located within the bulk of a finite density plasma background. In this case the relevant boundary condition on the charged fields states, that at large distances their densities approach a non-vanishing constant value ρ_0 which is the density of the uniform background plasma,

$$|\psi_{e,i}|^2 \xrightarrow{r \rightarrow \infty} \rho_0^2 \quad (27)$$

The canonical scale transformation assigns a non-trivial scale dimension to the charged fields. Consequently it can not leave the asymptotic particle density intact, and fails to be consistent with the boundary condition (27) unless $\rho_0 = 0$. Instead of the canonical version of the scale transformation, we need to employ a non-canonical version of (12), (13) where the scale dimensions of the charged fields vanish, $D_\psi = 0$. When we perform the ensuing variation of the fields in the energy density (18), instead of (26) we find

$$\delta_S \int d^3x T^0_0 = \int d^3x \left[\frac{E^2}{2} + \frac{B^2}{2} - \frac{1}{2\mu} \left(\sin^2 \alpha |D_k \psi_e|^2 + \cos^2 \alpha |D_k^* \psi_i|^2 \right) \right] \quad (28)$$

Now the integrand acquires both positive and negative contributions, which implies that a virial argument can not exclude the existence of stable finite energy solitons. Indeed, in [3] it has been argued that stable knotted solitons are present. These solitons are formed within the bulk of the plasma, in an environment with an asymptotically constant background density. A physical example of such an environment is the solar photosphere, the solitons are natural candidates for describing stable coronal loops. Another, somewhat more hypothetical example could be the ball lightning, in the background of Earth's atmosphere. Such solitons could also become relevant in identifying particularly stable plasma configurations in fusion experiments, when the plasma is kept at finite density by the boundaries of an appropriate vessel.

We shall now proceed to demonstrate, that the virial theorem (28) is also consistent with an appropriate canonical scale transformation. For this we first notice that excluding the kinetic terms, the Lagrangian (17) coincides with that of relativistic scalar electrodynamics with two flavors of scalar fields,

$$\mathcal{L} = |(\partial_\mu + iA_\mu)\phi_1|^2 + |(\partial_\mu - iA_\mu)\phi_2|^2 - V(\phi) - \frac{1}{4}F_{\mu\nu}^2 \quad (29)$$

Here we have included a Higgs potential $V(\phi)$, to ensure a non-vanishing asymptotic value for the charged fields. For example, we can choose $V(\phi) \propto (\phi_1^2 + \phi_2^2 - \rho_0^2)^2$. The

Hamiltonian version of (29) is

$$\begin{aligned} \mathcal{L} = & \pi_1^* \partial_0 \phi_1 + \pi_1 \partial_0 \phi_1^* + \pi_2^* \partial_0 \phi_2 + \pi_2 \partial_0 \phi_2^* + E_i \partial_0 A_i - |(\partial_k + iA_k)\phi_1|^2 - |(\partial_k - iA_k)\phi_2|^2 \\ & - \pi_1^* \pi_1 - \pi_2^* \pi_2 - V(\phi) - \frac{E^2}{2} - \frac{B^2}{2} - A_0 (\partial_i E_i + i\pi_1^* \phi_1 - i\pi_1 \phi_1^* - i\pi_2^* \phi_2 + i\pi_2 \phi_2^*) \end{aligned} \quad (30)$$

Notice that now the charged fields are canonically independent variables, a consequence of Lorentz invariance. The energy-momentum tensor can be computed directly from (10). With a proper symmetrization it becomes fully symmetric, as it should since the theory is Lorentz invariant. For the energy density we find

$$\begin{aligned} T^0_0 = & |D_k \phi_1|^2 + |D_k^* \phi_2|^2 + \frac{E^2}{2} + \frac{B^2}{2} + \pi_1^* \pi_1 + \pi_2^* \pi_2 + V(\phi) \\ & - A_0 \{ \partial_i E_i + i(\pi_1^* \phi_1 - \pi_1 \phi_1^*) - i(\pi_2^* \phi_2 - \pi_2 \phi_2^*) \} \end{aligned} \quad (31)$$

The momentum flux is

$$T^0_k = E_i F_{ki} + \pi_1^* D_k \phi_1 + \pi_1 D_k^* \phi_1^* + \pi_2^* D_k^* \phi_2 + \pi_2 D_k \phi_2^* \quad (32)$$

so that instead of (23), we find that the canonical scale dimensions of the charged scalar fields now vanish. As a consequence the canonical scale transformation is consistent with the relevant boundary condition that in the $r \rightarrow \infty$ limit the system approaches a Higgs vacuum $\phi_1^2 + \phi_2^2 \rightarrow \rho_0^2$. This means that the canonical scale transformation must now leave the energy of a static stationary point intact which leads to the following virial theorem

$$\begin{aligned} 0 = & \delta_S^C \int d^3x T^0_0 = - \int d^3x T^i_i \\ = & \int d^3x \left[\frac{E^2}{2} + \frac{B^2}{2} + 3(\pi_1^* \pi_1 + \pi_2^* \pi_2) - |D_k \phi_1|^2 - |D_k^* \phi_2|^2 \right] \end{aligned} \quad (33)$$

Since the contribution to the pressure from the charged fields is negative, the virial theorem can not exclude stable static solitons.

Finally, we note that even though the static sectors of the two theories (17) and (29) are very similar, these theories actually have a quite different physical content: In the relativistic case we may consistently set $\pi_1 = \pi_2 = E_i = A_0 = 0$ in the static equations of motion. This reduces the energy density to a functional form which is manifestly magnetohydrodynamical (2),

$$E = \int d^3x \left[|D_k \phi_1|^2 + |D_k^* \phi_2|^2 + V(\phi) + \frac{B^2}{2} \right] \quad (34)$$

But since the canonical scaling dimensions of the charged fields now vanish, the virial theorem does not exclude the existence of purely magnetic solitons. On the other hand,

if in the non-relativistic case (17) we set the electric field to vanish, the equations of motion become inconsistent unless the electron and ion charge densities are everywhere identical. This leads to a contradiction whenever the Hopf invariant is nontrivial [3], [4]. Hence solitons with a nontrivial Hopf invariant are necessarily accompanied with a nontrivial electric field. In particular, this means that their properties can not be consistently inspected by pure magnetohydrodynamics.

In conclusion, we have extended Shafranov's virial theorem from ideal magnetohydrodynamics to classical field theory and related it with Derrick's scaling argument. We then employed the virial theorem to inspect soliton stability in a realistic field theory model of a two component plasma. In line with ideal magnetohydrodynamics, a scaling argument reveals that the field theory model does not support stable isolated solitons in an otherwise empty space. But the virial theorem does allow for the existence of stable solitons within the bulk of the plasma. These solitons are accompanied by a nontrivial electric field, hence they can not be probed by magnetohydrodynamics alone. We suggest that these solitons are relevant in describing coronal loops in solar photosphere, maybe even ball lightning in Earth's atmosphere. They might also become useful in the design of particularly stable magnetic fusion geometries.

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