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Scenario for the renormalization in the 4D Yang−Mills theory*

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The renormalizability of the Yang–Mills quantum field theory in four-dimensional space–time is discussed in the background field formalism.

Yang–Mills quantum field theory has a unique character, allowing a self-consistent formulation in the four-dimensional space–time. Two important properties — asymptotic freedom and dimensional transmutation — which are the characteristic features that distinguish it from other theories. I think that the typical textbook exposition of this theory, which based on general paradigm of QFT, still does not underline these specifics. In my talk, I propose a scheme for the description of the Yang–Mills theory which exactly does this. I do not claim finding anything new; my proposal has simply a methodological value.

As a main tool for my presentation, I have chosen the object called the effective action which is defined via the background field. Following Feynman's ideas, I consider the functional of a background field as the generating functional for the S-matrix, whereas the Schwinger functional of external current, generating the Green functions, needs LSZ reduction formulas to define the S-matrix. Moreover, the latter functional is not manifestly gauge invariant.

Ironically, the standard description of the background field method[1–4] uses the external current and Legendre transformation. Alternative formulation began in Ref. 5 and entering Ref. 6 was improved in Ref. 7. In what follows I use the latter approach.

An effective action $W(B)$ is a functional of a classical Yang–Mills field $B_\mu(x)$ given by a series in a dimensionless coupling α,

$$W(B) = \frac{1}{\alpha} W_{-1}(B) + W_0(B) + \sum_{n\geq 1} \alpha^n W_n(B)\,,$$

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where $W_{-1}(B)$ is a classical action, $W_0(B)$ is a one-loop correction, defined via the determinants of vector and scalar operators M_1 and M_0,

$$M_0 = \nabla_\mu^2 , \quad M_1 = \nabla_\sigma^2 \delta_{\mu\nu} + 2[F_{\mu\nu}, \cdot] ,$$

where

$$\nabla_\mu = \partial_\mu + B_\mu ,$$
$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] ,$$
$$W_0 = -\frac{1}{2} \ln \det M_1 + \ln \det M_0$$

and W_k, $k = 1, 2, \ldots$ are defined as the contribution of strongly connected vacuum diagrams with $k+1$ loops, constructed via Green functions M_0^{-1} and M_1^{-1} and vertices defined by the forms of vector and scalar fields $a_\mu(x)$, $\bar{c}(x)$, $c(x)$

$$\Gamma_3(B) = g \int \operatorname{tr} \nabla_\mu a_\nu [a_\mu, a_\nu] d^4x ,$$
$$\Gamma_4(B) = g^2 \int \operatorname{tr} [a_\mu, a_\nu]^2 \, d^4x ,$$
$$\Omega(B) = g \int \operatorname{tr} \nabla_\mu \bar{c} [a_\mu, c] d^4x ,$$

where $g = \frac{1}{2}\sqrt{\alpha}$, taking into account anticommuting properties of ghosts $\bar{c}(x)$, $c(x)$.

The divergences of the diagrams should be regularized. I believe that there exists a regularization defined by the cutoff momentum Λ such that all infinities are powers in

$$L = \ln \frac{\Lambda}{\mu} ,$$

where μ is some normalization mass. Unfortunately, at present I do not know a satisfactory procedure for such a regularization. That is why I call my exposition a "scenario."

The renormalizability of the Yang–Mills theory means there exists a dependence of the coupling constant α on cutoff Λ such that the full action $W(B)$ is finite.

In the case of one loop, everything is clear. The functional $W_0(B)$ can be defined via the proper time method of Fock[8] giving the formula

$$W_0(B) = \int_0^\infty \frac{ds}{s} T(B, s) ,$$

where the functional $T(B, s)$ has the following behavior for small s,

$$T(B, s) = T_0(B) + sT_1(B, s)$$

and

$$T_0(B) = \frac{1}{2} \beta_1 W_{-1} ,$$

where β_1 is a famous negative constant. So the only divergence is proportional to the classical action and can be compensated by the renormalization of the coupling constant α. In more detail, we regularize $W_0(B)$ as

$$W_0^{\rm reg}(B) = \int_0^{1/\Lambda^2} ds\, T_1(B,s) + \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} T(B,s)$$

and choose the dependence of the coupling constant α on Λ as

$$\frac{1}{\alpha(\Lambda)} = -\beta_1 \ln \frac{\Lambda}{m}\,,$$

where m is a new parameter with the dimension of mass. It is clear that the regularized one-loop $W(B)$ does not depend on Λ and so is finite. More explicitly, we rewrite $W_0^{\rm reg}(B)$ as

$$W_{00}(B) = W_{00} + W_{01} L\,,$$

where

$$W_0^{\rm reg}(B) = \int_0^{1/\mu^2} ds\, T_1(B,s) + \int_{1/\mu^2}^{\infty} \frac{ds}{s} T(B,s)\,,$$

and

$$W_{01} = \beta_1 W_{-1}$$

and define the renormalized running coupling constant

$$\frac{1}{\alpha(\mu)} = -\beta_1 \ln \frac{\mu}{m}\,,$$

so that

$$W_{\text{1-loop}}^{\rm reg}(B) = \frac{1}{\alpha(\mu)} W_{-1} + W_{00}\,.$$

Thus we have traded the dimensionless α for a dimensional parameter m. However, m enters trivially, defining the scale. Observe that $\alpha(\Lambda)$ and $\alpha(\mu)$ are the values of the same function for two values of the argument. This function satisfies the first approximation to Gell-Mann–Low equation,

$$x \frac{d}{dx} \alpha(x) = \beta(\alpha) = \beta_1 \alpha^2\,,$$

where the r.h.s. does not depend on x and m plays the role of the conserved integral. The shift of x can be interpreted as an Abelian group action, this is the famous renormalization group. The same is true for the whole one-loop functional

$$W_{\text{1-loop}}^{\rm reg}(B,\Lambda) = W_{\text{1-loop}}(B,\mu)$$

and the renormalized action $W(B,\mu)$ does not depend on the running momentum μ.

My scenario is based on the assumption that the whole $W(B)$ has the same properties. I believe that all infinities in $W(B)$ can be combined into the form

$$W(B,\Lambda) = \frac{1}{\alpha}W_{-1} + W_{00} + W_{01}L + \alpha\big(W_{10} + W_{11}L\big) + \cdots + \alpha^n\big(W_{n0} + W_{n1}L + \cdots W_{nn}L^n\big) + \cdots\,,$$

where the functionals $W_{k0}(B)$, $k = 0, \ldots$ are finite and depend on μ. The coupling constant $\alpha(\Lambda)$ should satisfy the full Gell-Mann–Low equation

$$\Lambda\frac{d\alpha}{d\Lambda} = \beta(\alpha) = \beta_1\alpha^2 + \beta_2\alpha^3 + \cdots + \beta_n\alpha^n + \cdots$$

and the full $W(B)$ should be independent of L and so it is finite.

The equation

$$\frac{dW}{dL} = 0$$

immediately gives

$$W_{11} = \beta_2 W_{-1}$$

and leads to an equation expressed via double series in powers of α and L. The condition that the corresponding coefficients vanish leads to the recurrent relations expressing W_{nm} via $W_{n-1,m}$, $n = 2, 3, \ldots$, $n \geq m$.

Here are examples of such equations at the lowest orders:

$$\beta_1 W_{10} + W_{21} = \beta_3 W_{-1}\,,$$

$$\beta_2 W_{10} + 2\beta_1 W_{20} + W_{31} = \beta_4 W_{-1}\,,$$

$$\beta_1 W_{11} + 2W_{22} = 0\,,$$

$$\beta_2 W_{11} + 2\beta_1 W_{21} + 2W_{32} = 0\,.$$

One can solve some of these equations exactly. For instance, for the highest coefficients W_{nn} we get relation

$$(n-1)\beta_1 W_{n-1,n-1} + nW_{nn} = 0$$

and as W_{11} is proportional to W_{-1} the same is true for all W_{nn}. Thus their contribution $\sum \alpha^n L^n W_{nn}$ is summed up to

$$\frac{\beta_2}{\beta_1}\ln(1 + \beta_1\alpha L)W_{-1}$$

and this gives the next correction to the coefficient in front of W_{-1} in $W(B)$, namely

$$\frac{1}{\alpha} + \beta_1 L + \frac{\beta_2}{\beta_1}\left[\,\ln\alpha + \ln\left(\frac{1}{\alpha} + \beta_1 L\right)\right],$$

leading to the next approximation for the renormalized coupling constant

$$\frac{1}{\alpha_r} = \frac{1}{\alpha} + \beta_1 L + \frac{\beta_2}{\beta_1} \ln L \,,$$

consistent with the Gell-Mann–Low equation.

We see that all coefficients W_{nm}, $n \geq m$, $n \geq 1$ are expressed via the finite ones W_{n0}. More detailed investigation of these equations[9] shows that the full expression for $W(B)$ can be rewritten in terms of W_{n0} and powers of renormalized coupling constant

$$W(B,\mu) = \frac{1}{\alpha_r} W_{-1} + W_{00} + \sum \alpha_r^n W_{n0} \,.$$

This is consistent with the equation

$$W_{\mathrm{reg}}(B,\Lambda) = W(B,\mu)$$

and

$$L|_{\Lambda=\mu} = 0 \,.$$

Let us comment that the recurrence relations make sense when the only nonzero coefficients in β-function are β_1 and β_2. This enables a speculation that there exist the regularizations in which $\beta_3 = \beta_4 = \cdots = 0$.

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