Lecture Notes

L. D. Faddeev

Integrable Models, Quantum Groups and Conformal Field Theory*

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1 Introduction

The lectures by L.D. Faddeev where intended to give an introduction and survey to novel developments in quantum integrable systems, quantum groups and conformal field theories—of course with emphasis on the work done by the group at the Steklov Institute in St. Petersburg.

The notes follow closely the actual presentation of the material. So there is no intention to present full arguments and complete references in order to make the line of thought more transparent. The notes intend to give an idea of the actual speech—we hope you enjoyed it—as we have done as participants of the course.

We want to express our gratitude to L. D. Faddeev for his lecture and the many discussions about the material therein. Furthermore we should like to thank G. Haak, N. Kutz and M. Schmidt for their excellent and efficient writing and typing of the notes. This makes it possible to present them as the first preprint of the newly founded Sonderforschungsbereich "Differential Geometry and Quantum Physics".

Berlin, den 10. Dezember 1991, Ulrich Pinkall.

2 First lecture

The mathematical theory of solitons is about 25 years old. It started with the invention of the so-called inverse scattering method. The inverse scattering method was based on the introduction of the so-called linear auxiliary space, and the Lax equation, a very important additional idea was the Hamiltonian interpretation of these concepts. The Hamiltonian interpretation of the Korteweg-de Vries equation was first given by Gardner, Zakharov and Faddeev in 1971. In our approach we were mainly led by the idea of a future quantization of this subject, which was completely classical at that time. To quantize something one has to know first of all the Hamiltonian structure of the corresponding classical problem. Going step by step deeper into quantum mechanics an algebraic structure evolved, which was on one side very simple and on the other side quite universal. This algebraic structure will be a main topic in the following lectures.

Here we will consider only models of $1+1$ dimensional quantum field theory. For simplicity we investigate only systems with discrete space variable and continuous time variable, as in the Hamiltonian approach it is more natural not to discretize time.

Let $x = n \cdot \triangle$ be the space, t the time variable, with $n = 1 \dots N$ and $N+1 \equiv 1$, (one dimensional chain with periodic boundary conditions). With each site n we connect a Hilbert space h_n. The total space of physical states is thereby given by

$$\mathcal{H} = \bigotimes_{n=1}^{N} h_n. \tag{2.1}$$

We are given some dynamical algebra, generated by

$$X_n^a = \mathbb{1} \otimes \dots \otimes X^a \otimes \dots \otimes \mathbb{1}\,, \tag{2.2}$$

a some additional index, e.g. with respect to a Lie algebra basis, and all dynamical variables are required to be functions of the X_n^a.

Next impose the condition of ultralocality:

$$[X_n, X_m] = 0 \text{ for } n \neq m. \tag{2.3}$$

The dynamical equations are given as the usual Heisenberg equations:

$$\dot{X}_n^a = [H, X_n^a] \tag{2.4}$$

(a dot denoting derivation w.r.t. t). The idea of soliton theory is to associate to the Hamiltonian H a large series of commuting integrals of motion. To achieve this we introduce a new object, the so-called *Lax operator*

$$L_n(\lambda) = ((L_{n,ij}(\lambda)))_{m \times m}, \tag{2.5}$$

which is an $m \times m$ matrix in the auxiliary space $V = \mathbb{C}^m$ and with matrix entries which are operators on the Hilbert space h_n. The introduction of the additional parameter λ, called spectral parameter will become more transparent later.

We look for a nice instrument to display the various commutation relations between the matrix elements of the Lax operator in a compact way. By ultralocality these commutation relations do not depend on the index n. Skipping therefore the n-dependence we may express everything by terms of the form

$$L_{pq}(\lambda)L_{ij}(\mu) = (L(\lambda) \otimes L(\mu))_{pi|qj}. \tag{2.6}$$

and the products with reversed order of factors. Therefore it seems convenient to define the following operators on $V \otimes V$ constructed in terms of L:

$$L^1 = L \otimes \mathbb{1} \text{ and } L^2 = \mathbb{1} \otimes L \tag{2.7}$$

The commutation relations among the matrix elements may now be written as

$$R(\lambda - \mu)L^1(\lambda)L^2(\mu) = L^2(\mu)L^1(\lambda)R(\lambda - \mu) \tag{2.8}$$

with a matrix $R(\lambda) : V \otimes V \to V \otimes V$. We will call this relation the *fundamental commutation relations* (FCR). Considering L_n as a propagator we put now:

$$\psi_{n+1} = L_n \psi_n, \text{ where } \psi_n \in V. \tag{2.9}$$

This relation is called the *auxiliary problem.* It should be understood as a system of matrix equations with operator coordinates, i.e. as a system of equations in a noncommutative space.

If L_n for small $\triangle$ is of the form:

$$L_n = \mathbb{1} + \triangle \cdot L(x) + O(\triangle^2), \tag{2.10}$$

with $L(x) : V \to V$, the auxiliary problem becomes in the continuous limit

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = L(x)\psi. \tag{2.11}$$

Furthermore we have

$$\psi_{k+1} = L_k L_{k-1} \ldots L_1 \psi_1. \tag{2.12}$$

As *monodromy* it is natural to define

$$M \stackrel{\text{def}}{=} L_N \ldots L_1 \tag{2.13}$$

The matrix entries of M are now global operators on $\mathcal{H}$. The special feature of the FCR is, that the pure local relation (2.8) gives a global relation (global on H), for the monodromy M:

Lemma 2.1 *The FCR holds also for the monodromy M, i.e.:*

$$R(\lambda-\mu)M^1(\lambda)M^2(\mu) = M^2(\mu)M^1(\lambda)R(\lambda-\mu). \tag{2.14}$$

<u>Proof:</u> (In the following, we suppress the arguments λ and μ.)
By the FCR (2.8) and ultralocality we have for the products $L_k L_{k-1}$:

$$\begin{aligned} RL_k^1 L_{k-1}^1 L_k^2 L_{k-1}^2 &= RL_k^1 L_k^2 L_{k-1}^1 L_{k-1}^2 \\ &= L_k^2 L_k^1 R L_{k-1}^1 L_{k-1}^2 \\ &= L_k^2 L_k^1 L_{k-1}^2 L_{k-1}^1 R \\ &= L_k^2 L_{k-1}^2 L_k^1 L_{k-1}^1 R. \end{aligned} \tag{2.15}$$

The rest follows by induction.

The matrix entries of the monodromy are global operators. In order to get scalar commuting operators on the full Hilbert space $\mathcal{H}$ we take the trace of the monodromy.

$$F(\lambda) \stackrel{\text{def}}{=} \mathrm{tr} M(\lambda). \tag{2.16}$$

Assuming $R(\lambda)$ to be invertible and using (2.14) we get

$$F(\lambda)F(\mu) = F(\mu)F(\lambda). \tag{2.17}$$

The $F(\lambda)$ are therefore generators of an infinite dimensional algebra of commuting operators. We will return later to the generalization of the notion of complete integrability to infinite dimensional dynamical systems.

The second part of this lecture will illustrate the above framework by examples.

2.1 Example 1

Let $h_n := \mathbb{C}^2$ be the spin $\frac{1}{2}$ quantum Hilbert space and $\vec{S} = \frac{1}{2}\vec{\sigma}$ the spin operator where σ^a, $a = 1, 2, 3$, are the Pauli matrices. The S_n^a satisfy the commutation relations of $sl(2,\mathbb{C})$: $[S_n^a, S_m^b] = i\epsilon^{abc} S_n^c \delta_{nm}$. The Lax operator in our example is now:

$$L_n = \begin{pmatrix} \lambda + iS_n^3 & iS_n^+ \\ iS_n^- & \lambda - iS_n^3 \end{pmatrix} = \lambda \mathbb{1} \otimes \mathbb{1} + i\vec{S}_n \otimes \vec{\sigma} = \lambda \mathbb{1} \otimes \mathbb{1} + i \sum_{a=1}^{3} S_n^a \otimes \sigma^a.$$

So in this example $V = h_n$, which is not true in general. With the R-matrix given by:

$$R(\lambda) = \frac{1}{\lambda + i}(\lambda \mathbb{1} + iP) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $b = \frac{\lambda}{\lambda+i}$ and $c = \frac{i}{\lambda+i}$ the Lax operator L_n satisfies the FCR as may be verified by direct computation. As a Hamiltonian we choose the following element of the abelian algebra generated by the $F(\lambda)$:

$$H = \frac{\mathrm{d}}{\mathrm{d}\lambda}|_{\lambda=\frac{i}{2}} \ln F(\lambda) = \text{const.}(\sum_n S_n^a S_{n+1}^a + \text{const.}), \tag{2.18}$$

which turns out to be the Hamiltonian of the isotropic Heisenberg magnet. In this particular case the FCR holds for any spin representation, so that the example might be easily extended to higher spin j, with the corresponding Hilbert space $h_n = \mathbb{C}^{2j+1}$. In fact as was known for a long time the problem of complete integrability depends crucially on the representation of $sl(2,\mathbb{C})$. The Hamiltonian const.$(\sum_n S_n^a S_{n+1}^a + \text{const.})$ turns out not to have enough commuting integrals of motion for higher spin. But if we take the monodromy as given as in the case of spin $\frac{1}{2}$ we get $H = \frac{d}{d\lambda}|_{\lambda=\frac{i}{2}} \ln F(\lambda) = \sum_n f_j(S_n^a S_{n+1}^a)$, where the f's are polynomials of degree $2j$. These Hamiltonians turn out to be the "correct" generalizations of the spin $\frac{1}{2}$ case in the sense, that they posses "enough" integrals of motion to be completely integrable. Note, however, that the property of only pairwise interactions is conserved.

2.2 Example 2

The previous example will now be used to find in an analogous way a Lax operator and Hamiltonian for the nonlinear Schrödinger equation. For that purpose we construct an "infinite dimensional representation" for compact Lie groups in terms of oscillator variables. Starting from the usual commutation relations for annihilation and creation operators: $[\psi_n, \psi_m^\dagger] = \delta_{mn}$ we build up now spin operators, which are satisfying the $sl(2,\mathbb{C})$ commutation relations,

$$\begin{aligned} S_n^+ &= \psi_n^\dagger (2S - \psi_n^\dagger \psi_n)^{\frac{1}{2}}, \\ S_n^- &= (2S - \psi_n^\dagger \psi_n)^{\frac{1}{2}} \psi_n, \\ S_n^3 &= \psi_n^\dagger \psi_n - S. \end{aligned} \tag{2.19}$$

where S is any complex number. In the case where $2S$ is a positive integer one gets subrepresentations (just the Verma modules of $sl(2,\mathbb{C})$), which are finite dimensional. If S tends to infinity ("the quasiclassical limit"), we have the following behaviour of the spin operators in the vicinity of the lowest weight state (playing here the role of the ground state).

S_n^+ behaves like $\sqrt{S}\psi^\dagger$,
S_n^- behaves like $\sqrt{S}\psi$,
S_n^3 behaves like S.

With this one obtains the asymptotic behavior of the Lax operator (2.1),

$$\frac{1}{S}L_n = \begin{pmatrix} 1+\frac{\lambda}{S} & \frac{\psi_n^\dagger}{\sqrt{S}} \\ -\frac{\psi_n}{\sqrt{S}} & 1-\frac{\lambda}{S} \end{pmatrix} \sigma^3 = \left(\mathbb{1} + \triangle \begin{pmatrix} \lambda & \psi_n^\dagger(x) \\ \psi_n(x) & -\lambda \end{pmatrix} \right) \sigma_3 + O(\triangle^2), \quad (2.20)$$

if we put $S = \frac{1}{\triangle}$ and take into account, that

$$\psi_n = \triangle^{\frac{1}{2}}\psi(x). \quad (2.21)$$

The continuous quasiclassical limit of the Lax operator of the isotropic Heisenberg magnet is therefore nothing else than the well known Lax operator for the nonlinear Schrödinger equation.

2.3 Example 3

Another model might be obtained by substituting λ by $\sinh\lambda$, so we obtain for the Lax operator:

$$L_n = \begin{pmatrix} \sinh\lambda\cot\gamma + i\cosh(\lambda)S_n^3 & iS_n^+ \\ iS_n^- & \sinh\lambda\cot\gamma - i\cosh(\lambda)S_n^3 \end{pmatrix} \quad (2.22)$$

for the spin $\frac{1}{2}$ operator S_n^a. One realizes, that after the substitution $\frac{\lambda}{\gamma} \to \lambda$ one gets the same form of the Lax operator as in example (1) in the limit $\gamma = 0$. The R-matrix has the same form as in example (1) but the coefficients b, c are now substituted by:

$$b = \frac{\sinh\lambda}{\sinh(\lambda + i\gamma)} \text{ and } c = \frac{i\sin\gamma}{\sinh(\lambda+i\gamma)}. \quad (2.23)$$

The Hamiltonian may now be calculated to be

$$H = \text{const.}(S_n^1 S_{n+1}^1 + S_n^2 S_{n+1}^2 + \cos\gamma S_n^3 S_{n+1}^3), \quad (2.24)$$

which is the Hamiltonian of the anisotropic Heisenberg model (XXZ-model). The Lax operator may alternatively be written as

$$L_n(\lambda) = \frac{1}{\sin\gamma}\begin{pmatrix} \sinh(\lambda + i\gamma S_n^3) & i\sin\gamma S_n^+ \\ \sin\gamma S_n^- & \sinh(\lambda - i\gamma S_n^3) \end{pmatrix} \quad (2.25)$$

Unfortunately the fundamental commutation relations do not hold in the case of higher spin. But a slight modification leads again to fundamental commutation relations. Instead of the $sl(2,\mathbb{C})$ commutation relations, we impose the following deformed $sl(2,\mathbb{C})$ relation:

$$[S_n^3, S_n^\pm] = \pm S_n^\pm \text{ and } [S_n^+, S_n^-] = \frac{\sin(2\gamma S_n^3)}{\sin\gamma}. \quad (2.26)$$

If these relations are satisfied, then the $L_n(\lambda)$ of the form (2.25) satisfy the fundamental commutation relations but no longer form a Lie algebra. Instead they "generate" a new algebraic structure which gives in the limit $\gamma \to 0$ a Lie algebra. A realizations of the relations (2.26) by usual spin operators:

$$[\pi, \phi] = i\mathbb{1}$$

is given by:

$$S^{\pm} = \frac{1}{2S \sin \gamma} e^{\pm i\frac{\pi}{2}} (\mathbb{1} + 2S^2 \cos 2\phi)^{\frac{1}{2}} e^{\pm i\frac{\pi}{2}}, \ S^3 = \frac{\phi}{\gamma}.$$

Here now one obtains the quantum sine-gordon model as a special case of the Heisenberg XXZ-chain, which is connected to a Lie algebra deformation, which leads to the concept of a quantum group.

These examples cover the case of "rank 1", namely group $sl(2)$. One can use now groups of higher rank, there are generalizations of Lax operators for them and corresponding integrable models, the parameters in this list are: group, representation, anisotropy parameter (i.e. γ). It will be interesting to see if that is a classification.

3 Second lecture

In this lecture we want to look more carefully at the underlying algebraic structure which showed up in the last lecture. For a more abstract consideration we will suppress the dependence on the spectral parameter λ. Furthermore we change the notation for the generator matrices from L to T. So the general FCR reads now

$$RT^1T^2 = T^2T^1R, \tag{3.1}$$

with R acting as matrix on $V \otimes V$ and $T^1 = T \otimes \mathbb{1}$, $T^2 = \mathbb{1} \otimes T$ as previously defined.

Let $\mathcal{A}$ be the algebra, which is generated by the matrix entries of T, so that every element of $\mathcal{A}$ may be written as $f(T)$, where f is a polynomial in the matrix entries of T. Define a *comultiplication* on $\mathcal{A}$:

$$\triangle : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}, \tag{3.2}$$

on the generators of $\mathcal{A}$ by

$$\triangle(T_{ij}) = \sum_k T_{ik} \otimes T_{kj} \tag{3.3}$$

and extend it as an algebra homomorphism to the whole of $\mathcal{A}$. If $\tau \circ \triangle = \triangle$, where τ is the permutation $x \otimes y \rightarrow y \otimes x$ on $\mathcal{A} \otimes \mathcal{A}$, we call $\mathcal{A}$ *cocommutative.*

Example 3.1 Let $\mathcal{A} = \mathcal{C}(\mathcal{G})$, $\mathcal{G}$ a group, and define for all $f \in C(\mathcal{G})$

$$\triangle : C(\mathcal{G}) \rightarrow \mathcal{C}(\mathcal{G} \times \mathcal{G}) \cong \mathcal{C}(\mathcal{G}) \otimes \mathcal{C}(\mathcal{G})$$

as

$$(\triangle f)(g_1, g_2) = f(g_1 g_2).$$

This gives a comultiplication on the commutative algebra $\mathcal{A} = \mathcal{C}(\mathcal{G})$.

In an abstraction of the above example to the case of a noncommutative, noncocommutative algebra $\mathcal{A}$, Drinfeld called the resulting algebraic structure a *quantum group.*

Now the problem arises, if it is possible to find any nontrivial solutions to the FCR for a given R-matrix. We will see in the following, that we have to require certain conditions on R in order to get such solutions. Consider the operator $T^1T^2T^3$ acting as a matrix on $V \otimes V \otimes V$, where T^i acts as T on the i-th component of $V \otimes V \otimes V$ and as the identity on the others. There are two possibilities to interchange these three operators according to the paths:

$$\begin{array}{ccccc} & \nearrow & T^2T^1T^3 \rightarrow T^2T^3T^1 & \searrow & \\ T^1T^2T^3 & & & & T^3T^2T^1 \;. \\ & \searrow & T^1T^3T^2 \rightarrow T^3T^1T^2 & \nearrow & \end{array} \tag{3.4}$$

Define

$$R^{12}R^{13}R^{23} \equiv R^{123}$$

and

$$R^{23}R^{13}R^{12} \equiv R^{321}.$$

The realizaton of (3.4) gives:

$$\begin{aligned} T^1T^2T^3 &= (R^{123})^{-1}T^3T^2T^1R^{123} && (3.5)\\ &= (R^{321})^{-1}T^3T^2T^1R^{321}. && (3.6) \end{aligned}$$

In this way one obtains higher and higher relations on the matrix entries of T. But it turns out, that it is enough to require $R^{123} = R^{321}$, to get rid of all higher order relations simultaneously.

Definition 3.2 *The equation*

$$R^{12}(\lambda - \mu)R^{13}(\lambda)R^{23}(\mu) = R^{23}(\mu)R^{13}(\lambda)R^{12}(\lambda - \mu) \tag{3.7}$$

together with the FCR defines a quantized matrix algebra connected with the auxiliary space V. The equation (3.7) will be called Yang-Baxter equation.

One may look at the Yang-Baxter equation as a kind of Jacobi equation for the "structure constants" of the quantized matrix algebra. It appeared previously in statistical mechanics as well as in the theory of factorizable S-matrices.
In the previous lecture we defined the R-matrix:

$$R(\lambda) = \begin{pmatrix} \sinh(\lambda + i\gamma) & 0 & 0 & 0 \\ 0 & \sinh\lambda & i\sin\gamma & 0 \\ 0 & i\sin\gamma & \sinh\lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda + i\gamma) \end{pmatrix} \tag{3.8}$$

It is easy to check, that it satisfies the Yang-Baxter equation, which was to be expected, as we have already found a nontrivial solution L_n to the corresponding FCR. The quantum group, which is connected with $R(\lambda)$ is a kind of quantum loop group where λ is the loop parameter. To begin with the most simple examples of quantum groups, we want to get rid of the spectral parameter. This is possible, for example, by choosing the special point λ. The case $\lambda = 0$, which gives $R = R(0) = P$, the permutation matrix, is not interesting. Another possibiltity is to let λ tend to infinity. To get a well defined limit, we take instead of (3.8) the matrix

$$R(\lambda) = \begin{pmatrix} \sinh(\lambda + i\gamma) & 0 & 0 & 0 \\ 0 & \sinh\lambda & i\sin\gamma e^{\lambda} & 0 \\ 0 & i\sin\gamma e^{-\lambda} & \sinh\lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda + i\gamma), \end{pmatrix}, \tag{3.9}$$

which also satisfies the Yang-Baxter equation. Let $\lambda \to \infty$, then

$$R(\lambda) \stackrel{\lambda\to\infty}{\sim} \frac{e^{\lambda}}{2} \begin{pmatrix} e^{i\gamma} & 0 & 0 & 0 \\ 0 & 1 & e^{i\gamma} - e^{-i\gamma} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\gamma} \end{pmatrix}, \tag{3.10}$$

so that one obtains

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{3.11}$$

with $q := e^{i\gamma}$, the simplest example of an R-matrix which leads to a quantized matrix algebra of 2×2 matrices.

Let us now calculate the FCR for the 2×2 matrix:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.12}$$

The six nontrivial conditions resulting out of the FCR are:

$$\begin{aligned} ab &= q^{-1}ba, \\ dc &= qcd, \\ ad - da &= (q^{-1} - q)bc, \\ ac &= q^{-1}ca, \\ db &= qbd, \\ bc &= cb. \end{aligned} \tag{3.13}$$

The matrix algebra generated by the a, b, c, d together with these relations will be called $Gl_q(2, \mathbb{C})$. Here we think of the entries of a matrix as the generators of the polynomial agebra over the matrices, i.e. we rather quantize the algebra of functions over the Lie group in complete analogy with the first example. One may look at q as a deformation parameter, as one gets for $q = 1$ a commutative algebra, corresponding to usual matrices T.

Historically these relations were found by looking at the quantized Liouville model on the lattice.

A natural algebraic question which arises, is whether there exist central elements of the algebra $\mathcal{A}$. We find

$$det_q T \equiv ad - q^{-1}bc \tag{3.14}$$

which we may fix to be 1 to get a subalgebra, which we call $Sl_q(2, \mathbb{C})$. Another formal central element is $\frac{b}{c}$, which may be singular as c is not required to be invertible. T has an inverse with respect to matrix multiplication:

$$T^{-1} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} = S. \tag{3.15}$$

$T \to S = T^{-1}$ is therefore an antiautomorphism from $Sl_q(2,\mathbb{C})$ to $Sl_{\frac{1}{q}}(2,\mathbb{C})$, i.e. S satisfies the following fundamental commutation relations

$$RS^2S^1 = S^1S^2R. \tag{3.16}$$

We expect now, that there exist such quantum deformations for all classical groups. Corresponding R-matrices were found by Jimbo and Bazhanov.

As another example of a Hopf algebra, we look now at the algebra of functions on a Lie algebra $\mathbf{g}$, i.e. the universal enveloping algebra $U(\mathbf{g})$. Define a cocommutator on the generators of the algebra by

$$\triangle X = X \otimes \mathbb{1} + \mathbb{1} \otimes X \tag{3.17}$$

and extend it to the whole of U($\mathbf{g}$) as an algebra homomorphism. The resulting Hopf algebra is therefore cocommutative but not commutative.

By the identification with left-invariant vectorfields the elements $X \in U(\mathbf{g})$ may be considered as differential operators h(X) on C($\mathbf{g}$). In such a way we get a nondegenerate pairing between U($\mathbf{g}$) and $C(\mathcal{G})$

$$< X, f >= h(X)f(g)|_e. \tag{3.18}$$

So Lie algebra and Lie group are in a certain sense dual to each other.

In the following we will show, that the previously defined deformation of $sl(2)$ defined by the commutation relation

$$[S^+, S^-] = \frac{\sin(2\gamma S^3)}{\sin\gamma} \tag{3.19}$$

is dual to $Sl_q(2,\mathbb{C})$.

As there are no further nontrivial examples of quantum groups of 2×2 matrices with one generator matrix L satisfying the FCR, we consider now two matrices $L^{\pm}$ as generator matrices. In order to get not too many generators we restrict ourselves to triangular matrices, which corresponds to the choice of Borel subalgebras in a matrix Lie algebra.

$$L_+ = \begin{pmatrix} q^{\frac{H}{2}} & (q-q^{-1})X_+ \\ 0 & q^{\frac{-H}{2}} \end{pmatrix},\; L_- = \begin{pmatrix} q^{\frac{-H}{2}} & 0 \\ -(q-q^{-1})X_- & q^{\frac{H}{2}} \end{pmatrix}, \tag{3.20}$$

containing three generators $X_\pm$, H. The $L_\pm$ satisfy the following FCR's:

$$R^+ L^1_{\substack{+\\-\\+}} L^2_{\substack{+\\-\\-}} = L^2_{\substack{+\\-\\-}} L^1_{\substack{+\\-\\+}} R^+ \tag{3.21}$$

with $R^+ = PRP$, where P is the usual permutation matrix, so one has $R^+_{12} = R_{21}$, where $R_{21} = \sum_i R^i_2 \otimes R^i_1$ if $R_{12} = \sum_i R^i_1 \otimes R^i_2$. If we define $R^- \stackrel{\text{def}}{=} R^{-1}$ we get from (3.21) the remaining commutation relations,

$$R^- L^1_- L^2_+ = L^2_+ L^1_- R^-. \tag{3.22}$$

$R^{\pm}$ both satisfy the Yang-Baxter equations.

By direct computation of the FCR we get

$$q^{\frac{H}{2}} X_+ = q X_+ q^{\frac{H}{2}}, \tag{3.23}$$

$$X_+ X_- - X_- X_+ = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{3.24}$$

as defining relations on a new algebra $\mathcal{B}$, which is generated by $X_{\pm}$ and H. If one defines

$$< T^1, L^2_{\pm} > \stackrel{\text{def}}{=} R^{\pm} \tag{3.25}$$

one obtains a pairing of the Hopf algebras $\mathcal{A}$ and $\mathcal{B}$, which yields in the limit $\gamma \to 0$ ($q \to 1$) the above mentioned pairing between Lie group and universal enveloping algebra of the Lie algebra.

4 Third lecture

In the last lectures we saw that the concept of quadratic algebras defined by the FCR

$$RT^1T^2 = T^2T^1R. \tag{4.1}$$

with generators $T = ((T_{ij}))$, where T is a matrix on $V \otimes V$, V some auxiliary linear space, is relevant for defining quantum groups. We defined for example $Sl_q(2,\mathbb{C})$ by the FCR with R-matrix

$$R_t = \begin{pmatrix} t^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & t^{-\frac{1}{2}} & 0 & 0 \\ 0 & t^{\frac{1}{2}} - t^{-\frac{3}{2}} & t^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & t^{\frac{1}{2}} \end{pmatrix}, \tag{4.2}$$

where we used the normalization $R_t = t^{-\frac{1}{2}}R_q$, $q = t$ and took the freedom to transpose the matrix (which changes nothing essential).

In this context t plays the role of a deformation parameter, so that for $t = 1$ one obtains the usual matrix algebra. This lecture will deal mainly with the introduction of a differential geometric language in relation with quantum groups, which will result in some basic notions of noncommutative differential geometry. Here we look at the algebra generated by the T_{ij} as coordinates of the quantum group and try to enlarge the algebra by the introduction of a noncommutative analogon of differentials on the quantum group. Then we will investigate the additional commutation relations between them and the coordinates to get a generalization $(T^*\mathcal{G})_t$ of the cotangent bundle of a Lie group. We still deal with finite dimensional quantum groups and we will later associate this theory to conformal field theory.

The following mechanical picture will help us to get an intuition of the new concepts. Consider the isotropic spinning top with symmetry group $\mathcal{G} = SU(2)$. We generalize this example to an arbitrary Lie group $\mathcal{G}$. This generalized top has as phase space $T^*\mathcal{G}$. We want to quantize this example, and simultaneously produce a deformation of $\mathcal{G}$ in the sense of quantum groups.

Let $\omega \in \mathcal{G}$ and $a \in T^*_e\mathcal{G}$ be coordinates on the phase space $T^*\mathcal{G}$. The lagrangian is given in analogy to the case $\mathcal{G} = SU(2)$ by

$$l = \mathrm{tr}(\mathrm{d}\omega\omega^{-1}a) - \frac{1}{2}\mathrm{tr}\omega^2, \tag{4.3}$$

the first term being the canonical 1-form

$$\alpha = \mathrm{tr}(d\omega\,\omega^{-1}a). \tag{4.4}$$

The Poisson brackets of the coordinates are

$$\{\omega^1, \omega^2\} = 0 \tag{4.5}$$

$$\{a^1, \omega^2\} = C\omega^2 \tag{4.6}$$

$$\{a^1, a^2\} = \frac{1}{2}[C, a^1 - a^2], \tag{4.7}$$

where C is the Casimir operator of $\mathcal{G}$. For example for $\mathcal{G} = SU(2)$ we have

$$C = \sum_a \sigma^a \otimes \sigma^a \tag{4.8}$$

for the Casimir element, and it is easy to verify, that the usual Poisson structure for the top

$$\{a^\alpha, a^\beta\} = \epsilon^{\alpha\beta\gamma} a^\gamma, \tag{4.9}$$

may be given by the formula (4.7) if we set

$$a = \sum_\alpha a^\alpha \sigma^\alpha, \tag{4.10}$$

and using

$$\epsilon^{\alpha\beta\gamma} \sigma^\alpha \otimes \sigma^\beta = [C, \mathbb{1} \otimes \sigma^\gamma] \tag{4.11}$$

But we concentrate not only on the generalized velocities a (as it is normally done when studying the Poisson structure) but also on the coordinates.

This defines us a Poisson bracket, but says nothing about the direction in which we should try to deform it until we are given quadratic relations. So we would like to find a different parametrization of the phase space, which yields quadratic Poisson relations.

Restricting attention again to $\mathcal{G} = SU(2)$, in analogy with the chiral decomposition, we write

$$a = u_0 \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} u_0^{-1} \tag{4.12}$$

with $u_0 \in \mathcal{G}/\mathcal{H}$, $\mathcal{H}$ is the Cartan subalgebra of diagonal matrices in $\mathcal{G}$. We will treat u_0 and p in the following as independent variables. Next we introduce in addition to the left invariant differentials a their right invariant counterparts

$$\begin{aligned} a_L &\stackrel{\text{def}}{=} a, & (4.13)\\ a_R &\stackrel{\text{def}}{=} \omega^{-1} a_L \omega & (4.14)\\ &= v_0^{-1} \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} v_0, & (4.15) \end{aligned}$$

with $v_0 \in \mathcal{H}\backslash\mathcal{G}$. For getting a one-to-one parametrization, we require:

$$0 \leq p < \infty,$$

i.e. take into account the Weyl group. For the coordinates ω we set now

$$\omega = u_0 Q v_0, \tag{4.16}$$

where

$$Q = \begin{pmatrix} e^{iq} & 0 \\ 0 & e^{-iq} \end{pmatrix}. \tag{4.17}$$

If we parametrize u_0 and v_0 by Euler angles $\alpha, \beta, \gamma, \delta$

$$\begin{aligned} u_0 &= e^{i\alpha\sigma_3} e^{i\beta\sigma_2} && (4.18) \\ v_0 &= e^{i\gamma\sigma_2} e^{i\delta\sigma_3} && (4.19) \end{aligned}$$

We obtain a set of six variables $(\alpha, \beta, \gamma, \delta, p, q)$. The canonical 1-form (4.4) in the new coordinates reads now

$$\alpha = p\mathrm{d}q + p(\cos 2\beta \mathrm{d}\alpha + \cos 2\gamma \mathrm{d}\delta). \tag{4.20}$$

The variables p and $p \cos 2\beta$ ($p \cos 2\gamma$) may now be interpreted as full spin and third component of left (right) spin quantum numbers, as by definition p describes the $\mathcal{G}$-orbits.
The quantum Hilbert space is given by

$$\mathcal{H} = \sum_{\text{repr. of SU(2)}} \mathcal{H}_j \otimes \mathcal{H}_j, \tag{4.21}$$

where coordinates α, β are connected with the first and γ, δ with the second component.
Define now

$$u = u_0 Q, \ v = Q v_0, \tag{4.22}$$

i.e. add to u_0 and v_0 the missing Euler angles and set them equal to q. Then one has for u and v the nontrivial relations

$$\{u^1, u^2\} = u^1 u^2 r(p), \tag{4.23}$$

$$\{v^1, v^2\} = r(p) v^1 v^2, \tag{4.24}$$

with

$$r(p) = \frac{i}{p} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.25}$$

Thus we obtained quadratic relations for the other new variables.

Therefore we are now in a position to deform and quantize this classical example. But before we do this, let us review the main ideas of this procedure.

We had the FCR

$$RT^1T^2 = T^2T^1R$$

with R-matrix (4.2). Expanding $t = 1 + i\gamma\hbar + \ldots$, where γ is the deformation parameter and $\hbar$ Planck's constant (the deformation parameter of quantization), we get $R = \mathbb{1} + i\gamma\hbar r + \ldots$, with

$$r = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.26}$$

and $T^1T^2 - T^2T^1 + i\gamma\hbar(rT^1T^2 - T^2T^1r) + \ldots = 0$. With the usual Heisenberg quantization rule $\{,\} = \frac{i}{\hbar}[,]$ the last equation gives an additional Poisson structure on the classical limit of the quantum group

$$\{T^1, T^2\} = \gamma[r, T^1T^2]. \tag{4.27}$$

In the last equation T^1 and T^2 commute. This additional structure gives us now the direction, in which we may deform the group.

Taking care of the different form of our Poisson bracket relations (4.23) we look now for a deformation described by a fundamental commutation relation in the form

$$Ru^1u^2 = u^2u^1R(p). \tag{4.28}$$

Here now $R(p)$ is no longer a usual matrix with commutative entries. In fact the entries now depend on the variable p which is not central. Furthermore we require it to have a limit for $\hbar \to 0$, which leads us to the Poisson structure (4.23), i.e. $R = \mathbb{1} + i\gamma\hbar r + \ldots,\ R(p) = \mathbb{1} + i\gamma\hbar r(p) + \ldots$.

Similar to the old FCR the selfconsistency of (4.28) imposes certain conditions on $R(p)$ which we therefore call the generalized Yang-Baxter equations (GYBE)

$$R^{12}(p)(Q^2)^{-1}R^{13}(p)Q^2R^{23}(p) = (Q^1)^{-1}R^{23}(p)Q^1R^{13}(p)(Q^3)^{-1}R^{12}(p)Q^3. \tag{4.29}$$

To check this relation, remember that p and $Q = e^{iq\sigma_3}$ were introduced as conjugate variables. The commutation relations between p and q might be written as $e^{iq}p = (p - i\ln t)e^{iq}$.

From the history of integrable models we know already a natural candidate for $R(p)$, which was first given by Gervais and Neveu and deciphered by Bakelon and Faddeev to be the so-called quantized 6j-symbol of the rotation group.

$$R_t(p) = \begin{pmatrix} t^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & t^{\frac{1}{2}}\sqrt{1 - (\frac{t-t^{-1}}{e^{ip}-e^{-ip}})} & \frac{t^{\frac{1}{2}}-t^{-\frac{3}{2}}}{1-e^{2ip}} & 0 \\ 0 & \frac{t^{\frac{1}{2}}-t^{-\frac{3}{2}}}{1-e^{-2ip}} & t^{\frac{1}{2}}\sqrt{1 - (\frac{t-t^{-1}}{e^{ip}-e^{-ip}})} & 0 \\ 0 & 0 & 0 & t^{\frac{1}{2}} \end{pmatrix}, \text{ with } t = e^{i\gamma\hbar} \tag{4.30}$$

It depends only on e^{ip}, not on p alone, which makes it necessary to restrict the range of the coordinate p in the quantum case to the interval $0 \leq p < \pi$. Also for

$p \to -i\infty$ one has $R_t(p) \to R_t$, i.e. $R_t(p)$ contains R_t in a particular limit and satisfies the generalized YBE.

The classical limit is now achieved by the following choice

$$\hbar \to 0,\ \gamma \to 0 \tag{4.31}$$

and the renormalization

$$p \to \gamma p_{cl} \tag{4.32}$$

The last transformation, serves to restrict p to $0 \leq p_{cl} < \infty$. Now we have

$$\frac{t^{\frac{1}{2}} - t^{\frac{3}{2}}}{1 - e^{2ip}} \to \frac{\hbar}{p_{cl}} \text{ and } R(p) \to \mathbb{1} + i\gamma\hbar r + \ldots$$

The classical limit (4.31)–(4.32) gives the undeformed top.
We summarize:
The deformed and quantized top is described by

$$Ru^1u^2 = u^2u^1R(p),$$

$$R(p)v^1v^2 = v^2v^1R,$$

and further commutation relations between u_0, v_0 and p, q. The canonical coordinates of the cotangent bundle are defined by u, v, p and q as

$$\omega = uQ^{-1}v$$

and

$$A = u \begin{pmatrix} e^{ip} & 0 \\ 0 & e^{-ip} \end{pmatrix} u^{-1},$$

where A is introduced as an element of the deformed Lie algebra which is rather in the deformed group, i.e. we have in the limit

$$A = \mathbb{1} + \gamma a + \ldots$$

The resulting commutation relations may be computed to be

$$R^+\omega^1\omega^2 = \omega^2\omega^1R^+, \tag{4.33}$$

$$R^-\omega^1A^2 = A^2R^+\omega^1 \tag{4.34}$$

and

$$A^1(R^-)^{-1}A^2R^- = (R^+)^{-1}A^2R^+A^1, \tag{4.35}$$

where we have defined $R^+ = R$ and $R^- = PR^{-1}P$, P being the permutation matrix on $V \otimes V$.

Furthermore we have

$$R^\pm = \mathbb{1} + i\gamma\hbar r^\pm + \ldots,$$

with

$$r^+ = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 2 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \; r^- = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix},$$

such that $r^+ - r^- = C$. If we compute now the classical limit (4.31)–(4.32), we obtain from the relations (4.33)–(4.35) the classical commutation relations, which shows, how the classical theory is contained in our approach. In this sense one may call ω the generators of a quantized Lie group and A the generators of a quantized Lie algebra.

5 Fourth lecture

In this lecture, we will take a closer look on the derivation of the relations defining the quantum deformed top, which occurred at the end of the last lecture.

There we introduced the variables ω and A as an analogon of coordinates and momentum of the classical top. We had by analogy with the classical undeformed case $\omega = uQ^{-1}v$, with

$$Q = \begin{pmatrix} e^{iq} & 0 \\ 0 & e^{-iq} \end{pmatrix} \tag{5.1}$$

and $A_L = uDu^{-1}$, with

$$D = \begin{pmatrix} e^{ip} & 0 \\ 0 & e^{-ip} \end{pmatrix}, \tag{5.2}$$

with $[p, q] = i\gamma\hbar\mathbb{1}$ and $p = \gamma p_{cl}$. ¿From this we are now able to compute the commutators of Q with D

$$Q^1 D^2 = D^2 Q^1 t^\sigma \text{ with } \sigma = diag(1, -1, -1, 1). \tag{5.3}$$

The quadratic algebra is generated by u and v with relations

$$Ru^1u^2 = u^2u^1R(p) \tag{5.4}$$

and

$$R(p)v^1v^2 = v^2v^1R, \tag{5.5}$$

where $R(p)$ was given in the last lecture and satisfies the generalized YBE. Remember that $u_0 = uQ^{-1}$ and $v_0 = Q^{-1}v$ are commuting,

$$u_0^1v_0^2 = v_0^2u_0^1. \tag{5.6}$$

As p commutes with everything but q we see that

$$u_0^1D^2 = D^2u_0^1 \tag{5.7}$$

and by (5.3)

$$u^1D^2 = D^2u^1t^\sigma. \tag{5.8}$$

Clearly A_L doesn't depend on the diagonal matrix Q.

We want to write the commutation relations for ω and A:

$$\begin{aligned}
\omega^1\omega^2 &= u^1(Q^1)^{-1}v^1u^2(Q^2)^{-1}v^2 \\
&= u^1u_0^2v_0^1v^2 \\
&= u^1u^2(Q^2)^{-1}(Q^1)^{-1}v^1v^2 \\
&= R^{-1}u^2u^1R(p)(Q^2)^{-1}(Q^1)^{-1}(R(p))^{-1}v^2v^1R \\
&= R^{-1}u^2u^1(Q^1)^{-1}(Q^2)^{-1}v^2v^1R \\
&= R^{-1}u^2u_0^1v_0^2v^1R \\
&= R^{-1}u^2v_0^2u_0^1v^1R \\
&= R^{-1}\omega^2\omega^1R,
\end{aligned} \tag{5.9}$$

where we used the special form of Q^1Q^2:

$$Q^1Q^2 = \begin{pmatrix} e^{2iq} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2iq} \end{pmatrix} \tag{5.10}$$

and the fact that $R(p)$ has offdiagonal elements in the inner block only, so that Q^1Q^2 and $R(p)$ commute. Thus we derived the relation

$$R\omega^1\omega^2 = \omega^2\omega^1 R. \tag{5.11}$$

Next we investigate the relations between ω and A.

$$\begin{aligned} \omega^1 A^2 &= u^1 v_0^1 u^2 D^2 (u^2)^{-1} \\ &= u^1 u^2 D^2 (u^2)^{-1} v_0^1 \\ &= R^{-1} u^2 u^1 R(p) D^2 (u^2)^{-1} v_0^1 \\ &= R^{-1} u^2 D^2 (D^2)^{-1} u^1 R(p) D^2 (u^2)^{-1} v_0^1 \\ &= R^{-1} u^2 D^2 u^1 t^\sigma (D^2)^{-1} R(p) D^2 (u^2)^{-1} v_0^1. \end{aligned} \tag{5.12}$$

Now we use a property of $R(p)$ which will be taken here for granted and can be verified by direct computation (however, it's meaning will clarify in the last lecture)

$$t^\sigma (D^2)^{-1} R(p) D^2 = P(R(p))^{-1} P. \tag{5.13}$$

Using this and the fact that we may substitute $\tilde{R} \stackrel{\text{def}}{=} PR^{-1}P$ and $\tilde{R}(p) \stackrel{\text{def}}{=} PR(p)^{-1}P$ for R and $R(p)$ in the FCR (5.5) we have at last

$$R\omega^1 A^2 = A^2 \tilde{R} \omega^1. \tag{5.14}$$

If we define $R^- \stackrel{\text{def}}{=} R$, $R^+ \stackrel{\text{def}}{=} \tilde{R}$ we obtain the relations of the previous lecture:

$$R^-\omega^1 A^2 = A^2 R^+ \omega^1. \tag{5.15}$$

By the same way one may show

$$A^1 (R^-)^{-1} A^2 R^- = (R^+)^{-1} A^2 R^+ A^1. \tag{5.16}$$

So we get an algebra $(T^*\mathcal{G})_t$, containing coordinates and differentials. This algebra is not a Hopf algebra, but contains one, namely the quantum group G_t. One can use it for example to construct the exterior algebra $f(\omega, A^1, \ldots, A^n)$. Remember, however, that the variables A are more grouplike than in the classical case. This differs from the usual definition of cochains in algebraic topology.

In the classical limit

$$\gamma \to 0,\ \hbar \to 0,\ \{,\} = \frac{i}{\hbar}[,], \tag{5.17}$$

setting

$$A = \mathbb{1} + \gamma a + \ldots, \; R^{\pm} = \mathbb{1} + i\gamma\hbar r^{\pm} + \ldots \tag{5.18}$$

we get with the laplacian $C = r^+ - r^-$ and the classical top. With this we finish the discussion of the top and noncommutative geometry.

Now let us take a look at the theory of representation of quantum groups. Usually in physics we speak about representations, if some generators are realized as operators in a concrete space. This applies in our case to a Lie algebra. In the case of a group we have a different construction, which is to be called a corepresentation.

Indeed, when we speak of the representation of the classical group we consider a map

$$\begin{aligned} U : \mathcal{G} &\rightarrow Hom(V, K) \\ \mathbf{g} &\mapsto U(\mathbf{g}) \end{aligned} \tag{5.19}$$

to be a representation, where V is some vectorspace over the field K, if $U(\mathbf{g})$ satisfies the additional relation

$$U(\mathbf{g}_1\mathbf{g}_2) = U(\mathbf{g}_1)U(\mathbf{g}_2). \tag{5.20}$$

This is generalized to the case of quantum groups as follows. Let T be the generator matrix of a quantum group and consider the functions $U_{\alpha\beta}(t)$ (α, β in some index set), of its matrix elements t satisfying

$$U_{\alpha\beta}(\triangle t) = U_{\alpha\gamma}(t) \otimes U_{\gamma\beta}(t). \tag{5.21}$$

This defines us a corepresentation

$$\rho : t \mapsto \|U_{\alpha\beta}(t)\| \tag{5.22}$$

of $\mathcal{A}$. The fundamental commutation relations now depend on the representation.

$$R^{\rho}U^1U^2 = U^2U^1R^{\rho} \tag{5.23}$$

So it is the auxiliary space which changes when we consider different corepresentations but the quantum space remains the same. To summarize this we may write the following diagram:

	changes	doesn't change
corep. of coalgebra	auxiliary space	quantum space
rep. of algebra	quantum space	auxiliary space

After this classification let us consider representations and thus take the generators of an algebra, i.e. the generators $X_{\pm}$ and q^H on the quantum algebra $su_q(2)$. We had the following commutation relations (compare second lecture)

$$q^{\frac{H}{2}}X_+ = qX_+q^{\frac{H}{2}}, \; q^{\frac{H}{2}}X_- = q^{-1}X_-q^{\frac{H}{2}} \tag{5.24}$$

and

$$X_+X_- - X_-X_+ = \frac{q^H - q^{-H}}{q - q^{-1}}, \text{ where } q = e^{i\gamma},\ \gamma \text{ real }. \tag{5.25}$$

The unitarity conditions

$$X_+^* = X_-,\ H^* = H \tag{5.26}$$

are consistent with the above relations for all $q \in S^1 \in \mathbb{C}$. We choose an orthonormal basis $\{e_n\}$ of the representation space and inspired by the classical case we make the ansatz

$$X_+e_n = f(n)e_{n+1},\ X_-e_n = g(n)e_{n-1},\ q^{\frac{H}{2}}e_n = h(n)e_n. \tag{5.27}$$

where $f(n)$, $g(n)$, $h(n)$ are complex numbers. By (5.26) one has the relations

$$\overline{f}(n) = g(n+1),\ \overline{h}(n) = \frac{1}{h(n)}. \tag{5.28}$$

The commutation relations give us

$$h(n+1) = qh(n), \tag{5.29}$$

$$f(n-1)g(n) - g(n+1)f(n) = \frac{1}{q-q^{-1}}(h^2 - \frac{1}{h^2}), \tag{5.30}$$

or

$$|g(n)|^2 - |g(n+1)|^2 = \frac{1}{q - \frac{1}{q}}(h^2 - \frac{1}{h^2}). \tag{5.31}$$

Now the expression

$$A = (q+q^{-1})(q^H + q^{-H}) + (q-q^{-1})^2(X_+X_- + X_-X_+) \tag{5.32}$$

defines a central element—the deformation of the usual Casimir. We can write

$$Ae_n = (q-q^{-1})^2(|g(n)|^2) + |g(n+1)|^2) + (q+q^{-1})(h^2 + \frac{1}{h^2})e_n. \tag{5.33}$$

A solution to the above equations for $g(n)$ and $h(n)$ is

$$|g^2(n)| = \frac{1}{\sin^2\gamma}(\frac{1}{2}\cos(2n-1)\gamma + B) \tag{5.34}$$

and

$$h(n) = q^n. \tag{5.35}$$

where B is some constant. In order to obtain tracelessness of the finite dimensional representation matrices, n must be integer or semi-integer, consequently the spectrum of H is integer spaced and symmetric with respect to 0. The largest negative constant B, such that $|g^2|$ is still positive is $-\frac{1}{2}$. To obtain a finite dimensional representation one may therefore write

$$\begin{aligned} |g^2(n)| &= \frac{1}{\sin^2\gamma}\Big(\frac{1}{2}\cos((2n-1)\gamma) - \frac{1}{2}\cos(\underbrace{(2j+1)\gamma}_{p})\Big) \\ &= \frac{1}{\sin^2\gamma}\Big(\sin((n+j)\gamma)\sin((j-n+1)\gamma)\Big), \end{aligned} \tag{5.36}$$

with $0 \leq p < \pi$. Here $g(n) = 0$ for $n = -j$ and $g(n+1) = f(n) = 0$ for $n = j+1$, i.e. the highest and lowest weight vectors are e_j and e_{-j}, respectively.

Another possibility is to choose a positive $B \geq \frac{1}{2}$, then

$$g^2(n) = \frac{1}{\sin^2\gamma}\Big(\frac{1}{2}\cos(2n-1)\gamma + \frac{1}{2}\cosh\tilde{p}\Big), \text{ with } 0 \leq \tilde{p} < \infty. \tag{5.37}$$

We refer to this as the hyperbolic case. In this case one gets a finite dimensional representation only if $\gamma = \frac{\pi}{n}$, $n \in \mathbb{N}$. Then one has $g(n+N) = g(n)$ and imposing $e_{n+N} = e_n$, we get the so-called cyclic representation of dimension N.

6 Fifth lecture

In this and the next lecture we will investigate a new kind of symmetry, showing up in quantum mechanics which is described by quantum groups (confer the article of Mack and Schomerus or even more axiomatic Fredenhagen). Instead of our first toy model, the quantized top, we will now look at model, which is well known in quantum field theory, the Wess-Zumino-Novikov-Witten model (WZNW model).

Let $x_0 = x \in [0, 2\pi]$ and $x_1 = t \in \mathbb{R}$ be the space and time coordinates, respectively. The phase space is therefore a cylinder $M = S^1 \times \mathbb{R}$. We study fields $g(x,t)$ taking values in the Lie group $\mathcal{G} = SU(2)$. We introduce the left currents by pulling back the right invariant Maurer-Cartan form to M,

$$\mathcal{J}_\mu = \partial_\mu g g^{-1}, \tag{6.1}$$

which takes values in the corresponding Lie algebra, so one might use the Killing form to write as action

$$A = -\frac{1}{8\gamma}\int \mathrm{tr}\mathcal{J}_\mu^2 dx dt + \frac{1}{12\gamma}\int \left(\mathrm{d}^{-1}\mathrm{tr}((\mathrm{d}gg^{-1})^3)\right)_*. \tag{6.2}$$

The second term is called by the name of its inventors the Wess-Zumino (WZ) term. The $*$ here denotes pullback w.r.t. g, integration is over the whole space-time. The form $\mathrm{tr}((\mathrm{d}gg^{-1})^3)$ is not exact, the operation of taking the inverse of the derivation produces therefore singularities. The WZ term is therefore only defined up to an ambiguity. But this ambiguity plays no role in the functional integral, which contains only e^{iA}, if and only if it is integer valued. This forces

$$\gamma = \frac{\pi}{l},\ l \text{ an integer.} \tag{6.3}$$

The role of such multivalued actions in quantum field theory was investigated by Novikov.

We have therefore in the general WZNW model two parameters, a group $\mathcal{G}$ and an integer l, i.e. the same situation as for Kac-Moody algebras. We will see later that these two things are actually more intimately connected.

Let us investigate this model first classically and then quantize it. Consider g and $\mathcal{J}_0$ as independent variables and change the above lagrangian to

$$A = \frac{1}{4\gamma}\int (\mathcal{J}_0\partial_0 g g^{-1} - \frac{1}{2}\mathcal{J}_0^2 - \frac{1}{2}\mathcal{J}_1^2)\mathrm{d}x\mathrm{d}t + WZ\,\mathrm{term}, \tag{6.4}$$

from which we read off the canonical 1-form

$$\omega = \frac{1}{4\gamma}\int (\mathcal{J}_0\delta g g^{-1})\mathrm{d}x\mathrm{d}t + WZ\,\mathrm{term}. \tag{6.5}$$

δ denotes the differential in the phase space. Variation in the phase space gives the following symplectic form

$$\Omega = \delta\omega = \frac{1}{4\gamma} \int \mathrm{tr}\Big(\delta\mathcal{J}_0 \wedge \delta g g^{-1} + (\mathcal{J}_0 - \mathcal{J}_1)(\delta g g^{-1})^2\Big) \mathrm{d}x\mathrm{d}t, \tag{6.6}$$

because

$$\delta \mathrm{tr}((\mathrm{d}g g^{-1})^3) = 3\mathrm{d}\mathrm{tr}(\delta g g^{-1}(\mathrm{d}g g^{-1})^2). \tag{6.7}$$

The $\mathcal{J}_1$ term occurred because of the WZ term. In order to obtain the corresponding Poisson structure we invert the matrix (where δg is associated with the first row and column and $\delta\mathcal{J}_0$ with the second):

$$\begin{pmatrix} F & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & F \end{pmatrix}, \tag{6.8}$$

therefore the variables g commute,

$$\{g^1(x), g^2(y)\} = 0, \tag{6.9}$$

but

$$\{\mathcal{J}_0^1(x), g^2(y)\} = -2\gamma C g^2(y)\delta(x-y) \tag{6.10}$$

and

$$\{\mathcal{J}_0^1(x), \mathcal{J}_0^2(y)\} = \gamma[\mathcal{J}_0^1(x) - \mathcal{J}_0^2(y) - \mathcal{J}_1^1(y) + \mathcal{J}_1^2(y), C]\delta(x-y). \tag{6.11}$$

Furthermore we have

$$\begin{aligned}
&\{\mathcal{J}_0^1(x), \partial g^2(y)(g^2(y))^{-1}\} = \\
&= \; -2\gamma C(\partial g^2(y)\delta(x-y)(g^2(y))^{-1} + 2\gamma C g^2(y)\delta'(x-y)(g^2(y))^{-1} \\
&\quad + 2\gamma\partial g^2(y)(g^2(y))^{-1} C\delta(x-y),
\end{aligned} \tag{6.12}$$

$$\{\mathcal{J}_0^1(x), \mathcal{J}_1^2(y)\} = 2\gamma[C, \mathcal{J}_1^2(y)]\delta(x-y) + 2\gamma C\delta'(x-y). \tag{6.13}$$

If we define $L = \frac{1}{2}(\mathcal{J}_0 + \mathcal{J}_1)$ to be the left current w.r.t. light cone variables it follows

$$\{L^1(x), L^2(y)\} = \frac{1}{2}\gamma[C, L^1(x) - L^2(y)]\delta(x-y) + \gamma C\delta'(x-y) \tag{6.14}$$

and in coordinates w.r.t. the Pauli matrices, $L = L^a\sigma^a$,

$$\{L^a(x), L^b(y)\} = \gamma\epsilon^{abc}L^c\delta(x-y) + \gamma\delta^{ab}\delta'(x-y). \tag{6.15}$$

These are the commutation relations for a Kac-Moody algebra with central extension. A part of the phase space constitutes therefore a Kac-Moody algebra. Let us define a right current (which is left invariant) by

$$R = \frac{1}{2}g^{-1}(\mathcal{J}_0 - \mathcal{J}_1)g, \tag{6.16}$$

for which we have

$$\{R^1(x), R^2(y)\} = -\frac{1}{2}\gamma[C, L^1(x) - L^2(y)]\delta(x-y) - \gamma C\delta'(x-y) \tag{6.17}$$

and

$$\{L^1(x), R^2(y)\} = 0. \tag{6.18}$$

At last we have a chiral decomposition of the phase space in left and right movers, L and R respectively. However, it will be shown later on, that these variables are not enough to characterize the phase space completely. We are to introduce another scalar cyclic coordinate, which Poisson commutes with L and R. With this we have the coordinate transformation

$$(g, \mathcal{J}_0) \to (L(x), R(x), q) \tag{6.19}$$

and the hamiltonian H decouples into two currents:

$$H = \mathcal{J}_0^2 + \mathcal{J}_1^2 = L^2 + R^2. \tag{6.20}$$

The equations of motion are:

$$\begin{aligned} \partial_0 L + \partial_1 L &= 0 \text{ and} \qquad (6.21)\\ \partial_0 R - \partial_1 R &= 0. \qquad (6.22) \end{aligned}$$

Substituting the expressions for $\mathcal{J}_0$ and $\mathcal{J}_1$, (6.1), into L and R,

$$\begin{aligned} L &= \frac{1}{2}(\partial_0 g g^{-1} + \partial_x g g^{-1}) \qquad (6.23)\\ R &= \frac{1}{2}(g^{-1}\partial_0 g - g^{-1}\partial_x g) \qquad (6.24) \end{aligned}$$

and subtracting after multiplication by g from the right and left, respectively, we get for the old field $g(x,t)$

$$\partial_x g = Lg - gR. \tag{6.25}$$

As in the case of the top we make the ansatz

$$g = u(x)Kv(x), \tag{6.26}$$

where u and v satisfy $\partial_x u = Lu$ and $\partial_x v = -vR$ with initial conditions $u(0) = \mathbb{1}$, $v(0) = \mathbb{1}$, where we have chosen a fixed point $0 \in S^1$. We define the monodromy for u and v by

$$u(x+2\pi) = u(x)M_L, \text{ and } v(x+2\pi) = M_R v(x). \tag{6.27}$$

From the periodicity of $g(x)$ it follows that $K = M_L K M_R$. Therefore M_L and M_R^{-1} are conjugated by K and have the same spectrum. We set

$$M_L = Z_L D Z_L^{-1}, \; M_R = Z_R^{-1} D^{-1} Z_R, \tag{6.28}$$

where D is a diagonal matrix:

$$D = \begin{pmatrix} e^{ip} & 0 \\ 0 & e^{-ip} \end{pmatrix} \text{ with } 0 \le p \le \pi. \tag{6.29}$$

This implies that D and $Z_L^{-1} K Z_R^{-1}$ commute and the only freedom we have for K is

$$K = Z_L Q Z_R \text{ where } Q = \begin{pmatrix} e^{iq} & 0 \\ 0 & e^{-iq} \end{pmatrix}. \tag{6.30}$$

q is the looked for cyclic variable.

The above equations bear a strong resemblance to the equations for the quantized top, so that we may say, that the quantum top is hidden inside this model of conformal field theory.

L and R are chiral fields, so we obtain for the space and time dependence of u and v:

$$\begin{aligned} u(x,t) &= u(x-t), \\ v(x,t) &= v(x+t). \end{aligned} \tag{6.31}$$

Now introduce instead of u and v the Bloch-Floquet solutions

$$u_F(x) = u(x) Z_L Q, \; v_F(x) = Q Z_R v(x). \tag{6.32}$$

These are quasiperiodic as

$$u_F(x+2\pi) = u(x) M_L Z_L Q = u(x) Z_L D Q = u_F(x) D, \tag{6.33}$$

$$v_F(x+2\pi) = Q Z_R M_R v(x) = Q D^{-1} Z_R v(x) = D^{-1} v_F(x) \tag{6.34}$$

(Q and D commute). g is now given by

$$g = u_F Q^{-1} v_F. \tag{6.35}$$

In the next lecture we will quantize this example quite similar to the example of the top. It will be shown that we obtain a nice quadratic algebra in the variables u_F, v_F, p and q which may be quantized. The problem, which arises now, is that we have no longer ultralocality. Therefore we will first discretize the model in the x variable thereby returning to the methods of the first lecture, quantize on the lattice, and then study the continuous limit.

7 Sixth lecture

In this lecture we want to investigate further the connection between quantum groups and conformal field theory. The characteristic features, which identified the WZNW model of the last lecture as a conformal field theory, were the splitting of the algebra of observables in chiral components $\mathcal{A} = \mathcal{A}_L \otimes \mathcal{A}_R$ described by left and right movers L and R, respectively, and the action of the Virasoro algebra. The left mover L satisfied Kac-Moody algebra commutation relations, and the energy-momentum tensor, the generator of the Virasoro algebra is given by

$$T_L = \operatorname{tr}(L^2(x) + \partial_x L \sigma^3). \tag{7.1}$$

To circumvent the difficulties arising of the missing ultralocality of our Poisson bracket relations for the classical WZNW model we go to the lattice in order to find an appropriate quantization of the above model. Our main tool is again the construction of quadratic algebras as for the quantized top. We'll skip some of the standard calculations.

Let $x = n\triangle \to n$, $1 \leq n \leq N$ be the discrete space variable and L_n the discrete analogon of the left mover at site n.

Consider the commutation relations

$$\begin{aligned} R^+ L_n^1 L_n^2 &= L_n^2 L_n^1 (R^-)^{-1}, \\ L_n^1 L_{n+1}^2 &= L_{n+1}^2 R^+ L_n^1, \end{aligned} \tag{7.2}$$

and suppose, that L_n and L_m commute for $|n-m| \geq 2$. We can write these relations in a more short form

$$L_m^1 (R_{m-n-1}^-)^{-1} L_n^2 R_{m-n}^- = (R_{m-n}^+)^{-1} L_n^2 R_{m-n+1}^+ L_m^1, \tag{7.3}$$

where

$$R_n = \begin{cases} R & \text{for } n = 0 \\ \mathbb{1} & \text{for } n \neq 0 \end{cases} \tag{7.4}$$

In the continuous limit

$$R^\pm = \mathbb{1} + i\hbar\gamma r^\pm + \ldots, \tag{7.5}$$

$$L = \mathbb{1} + \triangle L(x) + \ldots, \tag{7.6}$$

$$\frac{\delta_{n,m}}{\triangle} \longrightarrow \delta(x-y), \ y = m\triangle \tag{7.7}$$

and

$$\frac{\delta_{n+1,m} - \delta_{n,m}}{\triangle^2} \longrightarrow \delta'(x-y). \tag{7.8}$$

and so get Poisson relations between $L(x)$ and $L(y)$ from the previous lecture taking into account, that $C = r^+ - r^-$.

We know already that the local field $g(x)$ is given by

$$g(x) = uKv, \tag{7.9}$$

Where u and v satisfy the auxiliary problems

$$u' = Lu, \; v' = -vR, \tag{7.10}$$

with $u(0) = v(0) = \mathbb{1}$. Now the discrete version reads

$$u_n = L_n u_{n-1}, \; u_0 = \mathbb{1} \tag{7.11}$$

or

$$u_n = L_n \ldots L_1. \tag{7.12}$$

and analogously for v; one has to substitute L with R and read the formulas from right to left. To get the commutation relations for the u_n's we therefore have to calculate the commutation relations of chains of L_i's. We first show, that L_n commutes with the triple $L_{n+1}L_nL_{n-1}$:

$$\begin{aligned} L_n^1 L_{n+1}^2 L_n^2 L_{n-1}^2 &= L_{n+1}^2 R^+ L_n^1 L_n^2 L_{n-1}^2 \\ &= L_{n+1}^2 L_n^2 L_n^1 (R^-)^{-1} L_{n-1}^2 \\ &= L_{n+1}^2 L_n^2 L_{n-1}^2 L_n^1. \end{aligned} \tag{7.13}$$

The commutation relations of u_n and u_m are therefore nearly independent of m and n. The situation becomes more complicated only if $n = m$ or $n = N$, the total number of sites in the lattice. We get

$$u_n^1 u_m^2 R^\pm = u_m^2 u_n^1 \begin{cases} + & \text{for } n > m \\ - & \text{for } n < m \end{cases} \tag{7.14}$$

and

$$R^+ u_n^1 u_n^2 = u_n^2 u_n^1 (R^-)^{-1}. \tag{7.15}$$

Similarly we obtain for v:

$$v_n^1 v_m^2 = R^\pm v_m^2 v_n^1 \begin{cases} + & \text{for } n > m \\ - & \text{for } n < m \end{cases} \tag{7.16}$$

and

$$(R^-)^{-1} v_n^1 v_n^2 = v_n^2 v_n^1 R^\pm, \tag{7.17}$$

while u_n and v_m commute for all m, n.

The element g has nontrivial commutation relations with L and R (as they all contain the left current $\mathcal{J}_0$, see the previous lecture), so that in the discrete version, setting $g_n = u_n K v_n$, we should require nontrivial commutation relations between K and L_1 and L_N,

$$\begin{aligned} \overset{1}{K}\overset{2}{L}_1 &= \overset{2}{L}_1 R^+ \overset{1}{K} \\ R^- \overset{1}{K}\overset{2}{L}_N &= \overset{2}{L}_N \overset{1}{K}, \end{aligned} \tag{7.18}$$

besides of the FCR

$$R\overset{1}{K}\overset{2}{K} = \overset{2}{K}\overset{1}{K}R. \tag{7.19}$$

We shall see soon, that these relations lead to the verification of periodic boundary conditions.

In the continuous limit the local field $g(x)$ is commutative:

$$\{g(x), g(y)\} = 0, \tag{7.20}$$

whereas we get from (7.16) and (7.19), that

$$\overset{1}{g}_n \overset{2}{g}_m = \overset{2}{g}_m \overset{1}{g}_n \tag{7.21}$$

$$R\overset{1}{g}_n \overset{2}{g}_n = \overset{2}{g}_n \overset{1}{g}_n R. \tag{7.22}$$

Let us define the left and right monodromy

$$u_N \overset{\text{def}}{=} M_L \text{ and } v_N \overset{\text{def}}{=} M_R \tag{7.23}$$

Periodic boundary conditions on the local field g should read

$$g_N = \underbrace{M_L K M_R}_{\overset{\text{def}}{=}\widetilde{K}} = g_0 = K. \tag{7.24}$$

One may show that $K(\widetilde{K})^{-1}$ commutes with every element of the algebra generated by the introduced coordinates and relations, which we always assume to be irreducible. This implies that $\widetilde{K}$ is K multiplied by a number which we believe to be 1.One of the reasons is the classical limit. At last we have for the monodromies

$$\begin{aligned} \overset{1}{M}_L (R^-)^{-1} \overset{2}{M}_L R^- &= (R^+)^{-1} \overset{2}{M}_L R \overset{1}{M}_L, \\ R^- \overset{1}{K} \overset{2}{M}_L &= \overset{2}{M}_L R^+ \overset{1}{K}. \end{aligned} \tag{7.25}$$

The comparision of equations (7.19) and (7.25) with the fourth lecture show us that there is a quantum top embedded in a highly nontrivial way inside our observable algebra.

So it turns out, that in each chiral component of a conformal invariant quantum field theory (as the WZNW model is in some sense universal) one has a hidden action of a (finite dimensional) quantum group, which is independent of the chosen regularization in the discretization procedure.

In the language of algebraic field theory, the monodromies M_L and M_R label the representations of the observable algebra and the K plays a role similar to that of a localized endomorphism between the superselection sectors.

Let us now look at the quasiperiodic Floquet solutions $u_F \stackrel{\text{def}}{=} uN_L$ and $v_F \stackrel{\text{def}}{=} N_R v$, which were already defined in the fifth lecture

$$K = Z_L Q Z_R \stackrel{\text{def}}{=} N_L Z_R \stackrel{\text{def}}{=} Z_L N_R. \tag{7.26}$$

N_L has the same commutation relations with left current L as K, as Z_R contains only the right current R. From the analogy with the top it follows, that N satisfy the relations

$$RN^1N^2 = N^2N^1R(p) \tag{7.27}$$

and comparising (7.17) and (7.27) we get

$$u_F^1(n)u_F^2(m) = u_F^2(m)u_F^1(n)R^{\pm}(p) \tag{7.28}$$

where $+$ stands for $n > m$ and $-$ for $n < m$. The mystical relation (5.13) which occurred in the fourth lecture is now nothing else than the consistency relation of the order on the lattice introduced by the relations (7.28) with periodicity of the x-space.

Let us write the total quantum Hilbert space $\mathcal{H}$ as sum of tensor products of representation spaces H_j^L of the Kac-Moody algebra generated by the left movers with some multiplicities B_j,

$$\mathcal{H} = \sum_j H_j^L \otimes B_j, \tag{7.29}$$

where

$$M_L : H_j^L \to H_j^L. \tag{7.30}$$

One may show that the center of the quantum Lie algebra and the center of the Kac-Moody algebra are equal, both are generated by the trace of the monodromy. Furthermore the representations of the quantum group and the Kac-Moody algebra are in one-to-one correspondence, we may therefore decompose the representation space H_j^L in a part V_j carrying a representation of the quantum group (zero modes) and an oscillator part H_0 independent of the spin label,

$$H_j^L = V_j \otimes H_0. \tag{7.31}$$

All this shows, that there are strong relations between Kac-Moody algebras and quantum groups. The level κ of the representation of a Kac-Moody algebra is explicitly given by $\kappa = l-2$, where l is connected with the deformation parameter γ of the quantum group via $\gamma = \frac{\pi}{l}$, $q = e^{i\gamma}$.

Taking now in account the second chiral component and the fact that the quantum top is embedded symmetrically in the chiral algebras we obtain

$$\mathcal{H} = \sum_j H_j^L \otimes H_j^R. \tag{7.32}$$

As was mentioned before the monodromies are operators on the representation spaces and K mediate between the different representations, in addition to (7.30) we get

$$K : H_j \to H_{j+\frac{1}{2}} \otimes H_{j-\frac{1}{2}} \tag{7.33}$$

and

$$M_R : H_j^R \to H_j^R. \tag{7.34}$$

In addition there is a gauge action of the quantum group on the observable algebra, i.e. for example

$$L_n \to h_{n+1} L_n h_n^{-1} \tag{7.35}$$

where the h_n are elements of a quantum group

$$R h_n^1 h_n^2 = h_n^2 h_n^1 R \tag{7.36}$$

commuting with L_n

$$h_n^1 L_m^2 = L_m^2 h_n^1. \tag{7.37}$$

We found a strong relationship between a finite dimensional quantum group $\mathcal{G}_q$ and an infinite dimensional Kac-Moody algebra KM_κ, both of which are parametrized by a compact group $\mathcal{G}$ and one additional parameter. KM_κ is the characteristic object in the description of $1+1$-dimensional conformal field theory. Recently N. Reshetikhin and I. Frenkel found a quantization $\mathrm{KM}_{\kappa,q}$ of Kac-Moody algebra KM_κ, which seems to be associated via a similar construction to the so-called Sklyanin algebra $\mathcal{G}_{q,u}$, which is a generalization of a quantum group and contains also two additional parameters. In short this may be summarized by the following diagram.

$$\begin{tikzcd}
 & & \mathcal{G} \arrow[dl] \arrow[dr] & \\
 & \mathrm{KM}_\kappa \arrow[rr, leftrightarrow] \arrow[dl] \arrow[d] & & \mathcal{G}_q \arrow[d] \\
(?) \arrow[r] & \mathrm{KM}_{\kappa,q} \arrow[rr, leftrightarrow] & & \mathcal{G}_{q,u}
\end{tikzcd}$$

The (?) may some day be filled with a Lie algebra connected with $2+1$-dimensional field theory, continuing the ladder to higher dimensions. One goes one step further in the dimension in the first line, because the WZNW model is a transgression of the topological field theory with the Chern-Simons lagrangian. If it is true also for the second line, than it follows, that there exists a four dimensional field theory which can be reduced to quantum mechanics. The search for such a model is an interesting and intriguing problem

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Lecture I

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