# Partially Dual Variables in SU(2) Yang-Mills Theory

Ludvig Faddeev*♯ and Antti J. Niemi**♯
*St.Petersburg Branch of Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27 , St.Petersburg, Russia
**Department of Theoretical Physics, Uppsala University, P.O. Box 803, S-75108, Uppsala, Sweden
♯Helsinki Institute of Physics, P.O. Box 9, FIN-00014 University of Helsinki, Finland
FADDEEV@PDMI.RAS.RU, NIEMI@TEORFYS.UU.SE



We propose a reformulation of SU(2) Yang-Mills theory in terms of new variables. These variables are appropriate for describing the theory in its infrared limit, and indicate that it admits knotlike configurations as stable solitons. As a consequence we arrive at a dual picture of the Yang-Mills theory where the short distance limit describes asymptotically free, massless point gluons and the large distance limit describes extended, massive knotlike solitons.

In the high energy limit Yang-Mills theory is asymptotically free, and can be solved perturbatively. It describes the interactions of massless gluons which correspond to the transverse polarizations of the gauge field $A_\mu$ [1].

At low energies Yang-Mills theory becomes strongly coupled. Perturbative techniques fail and nonperturbative methods must be developed. For this substantial efforts have been devoted, but numerical lattice approaches still remain the most viable tool to effectively explore the low energy theory. But in spite of our lacking theoretical understanding of low energy Yang-Mills theory, we expect that it exhibits color confinement with ensuing mass gap. The physical spectrum is supposed to describe massive composites of $A_\mu$ such as glueballs. When quarks are introduced the gauge field should form string-like flux tubes which confine quarks inside hadrons.

In the present Letter we propose an approach to investigate SU(2) Yang-Mills theory in the infrared limit. Our proposal is motivated by the qualitative picture developed in particular by 'tHooft and Polyakov [2], who asserted that the ultraviolet and infrared limits of a Yang-Mills theory represent different phases, with color confinement due to a dual Meissner effect in a condensate of magnetic monopoles. This picture suggests that even though the gauge field $A_\mu$ is the proper order parameter for describing the theory in its ultraviolet limit, in the infrared limit with monopole condensation some other order parameter could become more adequate. Naturally we expect, that such a change of variables may also imply certain need to reformulate the Yang-Mills action.

In the high energy limit the theory is described by the standard Yang-Mills action

$$S \; = \; \frac{1}{g^2} \int dx \; Tr F^2 \tag{1}$$

This is the *unique* Lorentz and gauge invariant local action which is renormalizable in four dimensions and admits a Hamiltonian interpretation which identifies the transverse polarizations of $A_\mu$ as the physical fields present in the ultraviolet limit.

In the following we shall propose new variables for describing the infrared limit of a four dimensional SU(2) Yang-Mills theory. We shall argue that instead of $A_\mu$, in this limit the appropriate order parameter is a three component vector $n^a(x)$ $(a = 1, 2, 3)$ with unit length $\mathbf{n} \cdot \mathbf{n} = 1$ and classical action [3]

$$S \; = \; \int dx \; m^2 (\partial_\mu \mathbf{n})^2 \; + \; \frac{1}{e^2} (\mathbf{n}, \mathbf{dn} \times \mathbf{dn})^2 \tag{2}$$

Here $m$ is a mass scale and $e$ is a dimensionless coupling constant. This is the *unique* local and Lorentz-invariant action for the unit vector $\mathbf{n}$ which is at most quadratic in time derivatives so that it admits a Hamiltonian interpretation, and involves *all* such terms that are either relevant or marginal in the infrared limit.

Observe that the action (2) can be related to the $SU(2)$ Skyrme model, restricted to a sphere $S^2$. However, the topological features of these two models are quite different.

We shall argue that (2) emerges from (1) by a change of variables together with a renormalization group argument. Thus it can be considered as a unique action for describing the low energy limit of a SU(2) Yang-Mills theory, in par with its high energy limit counterpart (1).

We note that in four dimensions the action (2) fails to be perturbatively renormalizable in the ultraviolet. But since it is expected to describe the physical excitations of a SU(2) Yang-Mills theory only in the low energy strong coupling limit, lack of perturbative renormalizability should not pose a problem provided we can interpret (2) adequately: In the following we shall argue that (2) can be derived from (1) by a renormalization group improved change of variables. Since (1) is renormalizable, this suggests that the quantum theory of (2) should also be consistent when properly treated. Indeed, we have recently established [4] that in 3+1 dimensions the classical action (2) describes stable knotlike solitons. This suggests that a proper route to its quantization should be based on the investigation of the quantum mechanical properties of these solitons.

From the point of view of a Yang-Mills theory the presence of knotlike solitons is actually quite appealing. It is natural to relate these solitons with the string-like flux tubes that we expect to be present in the infrared spectrum of a Yang-Mills theory, to provide the confining force between two quarks. In the absence of quarks such flux tubes may still be present as color-neutral excitations. They now close on themselves in knotted, stable solitonic configurations which are natural candidates for describing glueballs. In this manner we arrive at a dual picture of the Yang-Mills theory, with the high energy limit described by massless and pointlike transverse polarizations of $A_\mu$ and the low energy limit described by massive solitonic flux tubes which close on themselves in stable knotlike configurations

We shall now proceed to justify the action (2). We are motivated by the picture developed in [2], with confinement viewed as a dual Meissner effect in a condensate of magnetic monopoles. In a SU(2) Yang-Mills theory the relevant magnetic monopole is the (singular) Wu-Yang configuration [5]

$$A_i^a \;=\; \epsilon_{aik}\frac{x_k}{r^2} \tag{3}$$

and in order to describe a condensate of these monopoles, we need to properly extend (3) by introducing a smooth field for the corresponding order parameter. A natural Ansatz for extending (3) into a condensate is

$$A_i^a \;=\; \epsilon_{abc}\partial_i n^b n^c \;\equiv\; \mathbf{dn}\times\mathbf{n} \tag{4}$$

with $\mathbf{n}$ a three component unit vector field that describes the condensate. It reproduces (3) when we specify to the singular "hedgehog" configuration $\mathbf{n} = \mathbf{x}/r$.

The unit vector $\mathbf{n}$ describes two independent field variables. Since a gauge fixed four dimensional SU(2) connection $\mathbf{A}_\mu$ describes six polarization degrees of freedom, we need to extend the parametrization (4) by four additional polarizations. In order to search for a natural extension, we first observe that under an infinitesimal gauge transformation

$$\delta A_\mu^a \;=\; \nabla_\mu^{ab}\varepsilon^b \;=\; \partial_\mu\varepsilon^a + \epsilon^{acb}A_\mu^c\varepsilon^b$$

which is parametrized by the Lie algebra element $\varepsilon^a(x) = \varepsilon(x)\cdot n^a(x)$, (4) fails to remain form invariant. But if we improve (4) into

$$\mathbf{A}_\mu \;=\; C_\mu\mathbf{n} \;+\; \mathbf{dn}\times\mathbf{n} \tag{5}$$

where $C_\mu(x)$ is a vector field which transforms as an abelian connection

$$C_\mu \;\to\; C_\mu + \partial_\mu\varepsilon \tag{6}$$

the functional form of the configuration (5) remains intact under this gauge transformation.

The functional form (5) of SU(2) connections has been previously studied in particular by Cho [6], as a consistent truncation of the full four dimensional connection $A_\mu^a$. The goal in his work is to identify those field degrees of freedom in $A_\mu^a$ which are relevant for describing the Abelian dominance, a concept that originates from [2] and is expected to be relevant for color confinement.

The abelian gauge invariance (6) implies that (5) describes four field components, corresponding to the two transverse polarizations of the U(1) connection $C_\mu$ and the two independent components of $\mathbf{n}$. In order to extend (5) so that it describes all six field components of an arbitrary connection $\mathbf{A}_\mu$, we consider an arbitrary finite gauge transformation of a generic connection $A_\mu$. With

$$U(x) \;=\; \exp\{i\frac{1}{2}\alpha\mathbf{n}\cdot\tau\}$$

the SU(2) group element that determines this gauge transformation, we find for the gauge transformation of an arbitrary connection $A_\mu^a$

$$\mathbf{A}^U \;=\; [(\mathbf{A},\mathbf{n}) + \mathbf{d}\alpha]\mathbf{n} \;+\; \mathbf{dn}\times\mathbf{n}$$

$$+\;\sin\;\alpha\cdot(\mathbf{dn} + \mathbf{A}\times\mathbf{n}) - \cos\;\alpha\cdot(\mathbf{dn} + \mathbf{A}\times\mathbf{n})\times\mathbf{n} \tag{7}$$

From this we conclude that a generic connection $\mathbf{A}_\mu$ should have the functional form

$$\mathbf{A}_\mu \;=\; C_\mu\mathbf{n} \;+\; \mathbf{dn}\times\mathbf{n} \;+\; \varphi\mathbf{B}_\mu \;+\; \mathbf{B}_\mu\times\mathbf{n} \tag{8}$$

where $\varphi$ is a scalar field and $B_\mu^a$ is an orthogonal SU(2) valued vector, $\mathbf{n}\cdot\mathbf{B}_\mu = 0$ for all $\mu$. Since the number of independent field components carried by a four dimensional SU(2) connection is six, the orthogonal field $\mathbf{B}_\mu$ should only describe a single component, and we can select it to be proportional to $\mathbf{dn}$. This yields the following Ansatz for parametrizing a generic four dimensional connection,

$$\mathbf{A}_\mu \;=\; C_\mu\mathbf{n} \;+\; \mathbf{dn}\times\mathbf{n} \;+\; \rho\mathbf{dn} \;+\; \sigma\mathbf{dn}\times\mathbf{n} \tag{9}$$

Notice that we have here separated the second and fourth terms in the *r.h.s.*, even though these terms are linearly dependent. The reason for this separation is, that it allows us to combine the scalars $\rho$ and $\sigma$ into a complex field

$$\phi = \rho + i\sigma \tag{10}$$

with the property that under a SU(2) gauge transformation generated by $\alpha^a = \alpha\cdot\mathbf{n}$ the functional form of (9) remains intact, with the multiplet $(C_\mu, \phi)$ transforming like the field multiplet in the abelian Higgs model.

In order to verify that our parametrization (9) is indeed complete, we substitute it to the classical Yang-Mills action (1) and derive equations of motion obtained by varying the component fields $(\mathbf{n}, C_\mu, \phi)$. These equations should reproduce the original Yang-Mills equations, obtained by *first* varying *w.r.t.* $\mathbf{A}_\mu$ in (1) and then substituting (9):

If we introduce the U(1) covariant derivative

$$D_\mu \phi \ = \ \partial_\mu \phi \ + \ i C_\mu \phi$$

$$= \ \partial_\mu \rho \ - \ C_\mu \sigma \ + \ i \left( \partial_\mu \sigma \ + \ C_\mu \rho \right) \ = \ D_\mu \rho \ + \ i D_\mu \sigma \tag{11}$$

we find

$$\mathbf{F}_{\mu\nu} = \mathbf{n}(G_{\mu\nu} - [1 - (\rho^2 + \sigma^2)] H_{\mu\nu}) + (D_\mu \rho \, \partial_\nu \mathbf{n} - D_\nu \rho \, \partial_\mu \mathbf{n})$$

$$+ \ (D_\mu \sigma \ \partial_\nu \mathbf{n} \times \mathbf{n} \ - \ D_\nu \sigma \ \partial_\mu \mathbf{n} \times \mathbf{n}) \tag{12}$$

where

$$\begin{aligned} G_{\mu\nu} \ &= \ \partial_\nu C_\mu - \partial_\mu C_\nu \\ H_{\mu\nu} \ &= \ (\mathbf{n} \ , \ \partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}) \end{aligned}$$

When we substitute (12) into the Yang-Mills action (1) we get

$$S = \frac{1}{g^2} \int dx \Big\{ \ \mathbf{n} \left[ F_{\mu\nu} - (1 - [\rho^2 + \sigma^2]) H_{\mu\nu} \right] + (D_\mu \rho \ \partial_\nu \mathbf{n}$$

$$- \ D_\nu \rho \ \partial_\mu \mathbf{n}) \ + \ (D_\mu \sigma \partial_\nu \mathbf{n} \times \mathbf{n} - D_\nu \sigma \partial_\mu \mathbf{n} \times \mathbf{n}) \Big\}^2 \tag{13}$$

and when we perform the variations *w.r.t.* $(C_\mu, \phi, \mathbf{n})$ we get

$$\begin{aligned} \mathbf{n} \cdot \nabla_\mu \mathbf{F}_{\mu\nu} \ &= \ 0 \\ \partial_\nu \mathbf{n} \cdot \nabla_\mu \mathbf{F}_{\mu\nu} \ &= \ 0 \\ \partial_\nu \mathbf{n} \times \mathbf{n} \cdot \nabla_\mu \mathbf{F}_{\mu\nu} \ &= \ 0 \\ ( \ D_\nu \rho \ + \ D_\nu \sigma \cdot \mathbf{n} \times \ ) \cdot \nabla_\mu \mathbf{F}_{\mu\nu} \ &= \ 0 \end{aligned}$$

which are all proportional to the ordinary Yang-Mills equation, evaluated at the field (9). But the U(1) invariance (6) implies that only six of these equations can be independent. These equations coincide with the six independent second order equations that we obtain when we first vary the action (1) *w.r.t.* the full connection $A^a_\mu$ and then substitute for the parametrization (9). Thus we assert that the parametrization (9) is indeed complete. (We remind that the variation of (1) *w.r.t.* $A^a_\mu$ yields twelve equations, but the three $A^a_0$ are Lagrange multipliers and three of the equations are first order, corresponding to Gauss law in the Hamiltonian approach. Consequently in four dimensional SU(2) Yang-Mills theory there are only six independent second order equations.)

We observe that the second term in (2) is also present in (13). Since the first term in (2) involves a mass scale it is absent in (1), as there is no way to introduce a mass scale in four dimensional Yang-Mills by employing ultraviolet renormalizable, local, Lorentz and gauge invariant functionals of $A_\mu$. However, when we represent the Yang-Mills action using the component field (8), (9), there is nothing *a priori* that would prevent us from including the first term in (2) already at the tree level. Indeed, since it is a relevant operator in the infrared, even if absent at the tree level it should emerge when we account for quantum fluctuations in a gradient expansion. We assert that these fluctuations produce a non-vanishing expectation value when we average over the scalar field $\phi = \rho + i\sigma$ in (13) to the effect

$$< |\partial_\lambda \phi|^2 \ \eta_{\mu\nu} - \partial_\mu \phi^* \partial_\nu \phi > = \ m^2 \eta_{\mu\nu} \tag{14}$$

As a consequence we conclude that the full action (2) is contained in a gradient expansion of the effective action for the order parameter $\mathbf{n}$.

Obviously the full effective action for the order parameter $\mathbf{n}$ obtained by integrating over the complete set of fields in the parametrization (8), will also contain various additional functionals of $\mathbf{n}$ besides the two terms that appear in (2). However, (2) is *unique* in the sense that it contains *all* such infrared relevant and marginal, local Lorentz invariant operators of $\mathbf{n}$ which are at most quadratic in time derivatives, as is necessary for a Hamiltonian interpretation. Consequently we may as well adopt the point of view, that (2) is the *unique* fundamental action to describe the low energy limit of a SU(2) Yang-Mills theory, in the confining phase where magnetic monopoles condense. There is no alternative which would be consistent with our general principles! The results of [4] then suggest that at low energies the physical states of the Yang-Mills theory are knotlike solitons of the monopole condensate, and it becomes natural to view these configurations as candidates for describing glueballs.

Notice that the present interpretation of (2) is entirely analogous to the common point of view to consider (1) as the fundamental action for the high energy Yang-Mills theory, even though *e.g.* a gradient expansion of the lattice Yang-Mills action involves higher derivative terms which all become irrelevant in the continuum (short-distance) limit where the lattice spacing tends to zero.

Besides the order parameter **n** which is appropriate for describing the phase with monopole condensation, we have also found that the abelian Higgs multiplet $(C_\mu, \phi)$ naturally appears in the parametrization of four dimensional connections. Elimination of **n** in (13) then produces an effective action for the abelian Higgs multiplet which comprises a natural order parameter for describing the SU(2) theory in a "Higgs phase", also considered in [2]. Indeed, since we have the spontaneously broken Higgs self-coupling present in (13),

$$V(\phi) \ = < H_{\mu\nu}^2 > \cdot (1 - [\rho^2 + \sigma^2])^2 \ \sim \ \lambda(1 - |\phi|^2)^2$$

we can expect the corresponding effective action to support Nielsen-Olesen type vortices as infinite energy line solitons. In a sense, these abelian Higgs variables can be viewed as dual to the vector field **n** in the expansion (9), and it would be of interest to further study the properties of this "Higgs phase".

In conclusion, we have argued that (2) is the unique action for describing SU(2) Yang-Mills theory at low energies, consistent with various natural first principles. It can be derived directly from (1) by a renormalization group improved change of variables. The unit vector **n** can be viewed as an order parameter for monopole condensation, and the first term in (2) should be included since it is relevant in the infrared limit. This term introduces a mass gap and the ensuing action supports knotlike configurations as stable solitons. This suggests a dual picture of the Yang-Mills theory where the high energy limit describes massless pointlike gluons and the infrared limit describes massive knotted solitons, consistent with the commonly accepted picture of color confinement. Furthermore, we have found that the parametrization of a generic connection also contains the abelian Higgs multiplet, in a manner which can be viewed as dual to the vector field **n**. This suggests that line vortices of the abelian Higgs model may also be present in a description of the theory in an appropriate "Higgs phase", in line with the original picture in [2]. It would be interesting to further study the properties of this phase.

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