

PARTIAL DUALITY IN $SU(N)$ YANG-MILLS THEORY

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Recently we have proposed a set of variables for describing the infrared limit of four dimensional $SU(2)$ Yang-Mills theory. Here we extend these variables to the general case of four dimensional $SU(N)$ Yang-Mills theory. We find that the $SU(N)$ connection A_μ decomposes according to irreducible representations of $SO(N-1)$, and the curvature two form $F_{\mu\nu}$ is related to the symplectic Kirillov two forms that characterize irreducible representations of $SU(N)$. We propose a general class of nonlinear chiral models that may describe stable, soliton-like configurations with nontrivial topological numbers.

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Recently [1] we have proposed a novel decomposition of the four dimensional $SU(2)$ Yang-Mills connection A_μ^a . In addition of a three component unit vector n^a , it involves an abelian gauge field C_μ and a complex scalar $\phi = \rho + i\sigma$. The fields C_μ and ϕ determine a $U(1)$ multiplet under $SU(2)$ gauge transformations in the direction n^a ,

$$A_\mu^a = C_\mu n^a + \epsilon_{abc} \partial_\mu n^b n^c + \rho \partial_\mu n^a + \sigma \epsilon^{abc} \partial_\mu n^b n^c \quad (1)$$

In four dimensions this decomposition is complete in the sense that it reproduces the Yang-Mills equations of motion [1]. This is already suggested by the number of independent fields: if we account for the $U(1)$ invariance, (1) describes $D + 2$ field degrees of freedom. For $D = 4$ this equals $3(D - 2)$, the number of transverse polarization degrees of freedom described by a $SU(2)$ connection in D dimensions. Furthermore, if we properly specify the component fields, (1) reduces to several known $SU(2)$ field configurations. As an example, if we set $A_0 = C_\mu = \rho = \sigma = 0$ and specify n^a to coincide with the (singular) radial vector x^a/r , the parametrization (1) yields the (singular) Wu-Yang monopole configuration [2]. As another example, if we specify a rotation symmetric configuration with $C_i = Cx^i$, and C_0, C, ϕ to depend on r and t only, and set $n^a = x^a/r$, (1) reduces to Witten's Ansatz for instantons [3].

Here we wish to generalize (1) to $SU(N)$. We shall argue that exactly in four dimensions the $SU(N)$ Yang-Mills connection admits the following decomposition

$$A_\mu^a = C_\mu^i m_i^a + f^{abc} \partial_\mu m_i^b m_i^c + \rho^{ij} f^{abc} \partial_\mu m_i^b m_j^c + \sigma^{ij} d^{abc} \partial_\mu m_i^b m_j^c \quad (2)$$

With $i = 1, \dots, N - 1$ we label the Cartan subalgebra. We shall construct the $N - 1$ mutually orthogonal unit vector fields m_i^a (with $a = 1, \dots, N^2 - 1 = \text{Dim}[SU(n)]$) in the following) so that they describe $N(N - 1)$ independent variables. The combination

$$A_\mu^a = C_\mu^i m_i^a + f^{abc} \partial_\mu m_i^b m_i^c \quad (3)$$

is the $SU(N)$ Cho connection [4], under the $N - 1$ independent gauge transformations generated by the Lie-algebra elements

$$\alpha_i = \alpha_i m_i^a T^a \quad (4)$$

the vector fields C_μ^i transform as $U(1)$ connections

$$C_\mu^i \rightarrow C_\mu^i + \partial_\mu \alpha_i$$

Consequently these fields describe $(D - 2)(N^2 - 1)$ independent variables. The $\phi_{ij} = \rho_{ij} + i\sigma_{ij}$ are $N(N - 1)$ independent complex scalars, mapped onto each other by the $U(1)$ gauge transformations (4). As a consequence (C_μ^i, ϕ_{ij}) can be viewed as a collection of abelian Higgs multiplets. We shall find that the fields ρ_{ij} and σ_{ij} decompose according

to the traceless symmetric tensor, the antisymmetric tensor, the vector and the singlet representations of $SO(N-1)$. In D dimensions (2) then describes

$$N(N-1) + (D-2)(N-1) + N(N-1) = 2N^2 + (D-4)N + (2-D)$$

independent variables. In *exactly* $D = 4$ this coincides with $(D-2)(N^2-1)$ which is the number of independent transverse variables described by a $SU(N)$ Yang-Mills connection A_μ^a .

We also observe that for $D = 3$, the number of independent variables in (2) coincides with the dimension of the $SU(N)$ gauge orbit, *independently* of the Yang-Mills equations of motion.

We note that a decomposition of the $SU(N)$ connection has been recently considered by Periwal [5]. It appears that his results are different from ours.

We proceed to the justification of the decomposition (2). For this we need a number of group theoretical relations. Some of these relations seem to be novel, suggesting that further investigations of (2) could reveal hitherto unknown structures.

The defining representation of the $SU(N)$ Lie algebra consists of $N^2 - 1$ traceless hermitean $N \times N$ matrices T^a with

$$T^a T^b = \frac{1}{2N} \delta^{ab} + \frac{i}{2} f^{abc} T^c + \frac{1}{2} d^{abc} T^c$$

normalized to

$$(T^a, T^b) \equiv Tr(T^a T^b) = \frac{1}{2} \delta^{ab}$$

The f^{abc} are (completely antisymmetric and real) structure constants. The d^{abc} are the (completely symmetric) coefficients

$$\frac{1}{2} d^{abc} = Tr(T^a \{T^b, T^c\}) \equiv Tr(T^a (T^b T^c + T^c T^b))$$

and we note that in the defining representation we can select the Cartan subalgebra so that

$$\sum_{i,j=1}^{N-1} d^{ijk} d^{ijl} = \frac{2(N-2)}{N} \delta_{kl} \quad (5)$$

For any four matrices A, B, C, D we have

$$Tr([A, B] \{C, D\} + [A, C] \{B, D\} + [A, D] \{B, C\}) = 0$$

and

$$Tr([A, B][C, D] + \{A, C\} \{B, D\} - \{A, D\} \{B, C\}) = 0$$

Hence, if we introduce the matrices $(\mathcal{F}^a)^{bc} = f^{abc}$ (which define the adjoint representation) and $(\mathcal{D}^a)^{bc} = d^{abc}$, we have

$$[\mathcal{F}^a, \mathcal{D}^b] = -f^{abc} \mathcal{D}^c \quad (6)$$

and

$$[\mathcal{D}^a, \mathcal{D}^b]_{cd} = f^{abe} \mathcal{F}_{cd}^e + \frac{2}{N}(\delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}) \quad (7)$$

Note that the \mathcal{D}^a are traceless.

We conjugate the Cartan matrices H_i of the defining representation by a generic element $g \in SU(N)$. This produces set of Lie algebra valued vectors

$$m_i = m_i^a T^a = g H_i g^{-1} \quad (8)$$

an over-determined set of coordinates on the orbit $SU(N)/U(1)^{N-1}$: By construction the m_i^a depend on $N(N-1)$ independent variables, they are orthonormal,

$$(m_i, m_j) = m_i^a m_j^a = \delta_{ij}$$

and it is straightforward to verify that

$$[m_i, m_j] = 0 \quad (9)$$

$$\{m_i, m_j\} = d^{ijk} m_k \quad (10)$$

$$Tr(m_i \partial_\mu m_j) = (m_i, \partial_\mu m_j) = 0 \quad (11)$$

We can also show that

$$-f^{acd} m_i^c f^{deb} m_i^e = \delta^{ab} - m_i^a m_i^b \quad (12)$$

is a projection operator. This can be verified *e.g.* by using explicitly the defining representation of $SU(N)$. The result follows since the weights of $SU(N)$ have the same length and the angles between different weights are equal. Notice that (12) can also be represented covariantly, using the $SU(N)$ permutation operator

$$- [m_i, T^d] \otimes [m_i, T^d] = T^b \otimes T^a - m_i \otimes m_i \quad (13)$$

We now consider the Maurer-Cartan one-form

$$dgg^{-1} = \omega_{a\mu} T^a dx^\mu$$

We find

$$dm_i = [\omega, m_i] \quad (14)$$

We now recall that each Cartan matrix H_i can be used to construct a symplectic (Kirillov) two-form on the orbit $SU(N)/U(1)^{N-1}$,

$$\Omega^i = Tr(H_i [g^{-1} dg, g^{-1} dg]) \quad (15)$$

$$d\Omega^i = 0$$

The Ω^i are related to the representations of $SU(N)$; According to the Borel-Weil theorem each linear combination of Ω_i

$$\sum_i n_i \Omega^i \quad (16)$$

with integer coefficients corresponds to an irreducible representation of $SU(N)$.

When we combine the projection operator (12) with the relation (14) and the Jacobi identity, we find that these two-forms Ω^i can be represented in terms of the m_i ,

$$\Omega^i = Tr(m_i[dm_k, dm_k]) = (m_i, dm_k \wedge dm_k) \quad (17)$$

Explicitly, in components

$$\Omega_{i,\mu\nu} = f^{abc} m_i^a \partial_\mu m_k^b \partial_\nu m_k^c \quad (18)$$

Next, we proceed to consider a generic Lie algebra element $v = v^a T^a$. We define the (infinitesimal) adjoint action δ^i of the m_i on v by

$$\delta^i v = [v, m_i] \quad (19)$$

Using (12) and by summing over i we find that this yields (up to a sign) a projection operator to a subspace which is orthogonal to the maximal torus and is spanned by the m_i ,

$$(\delta^i)^2 v = -v + m_i(m_i, v) \quad (20)$$

We shall also need a (local) basis of Lie-algebra valued one-forms in the subspace to which (20) projects. For this we first use (10) to conclude that for the symmetric combination

$$\{dm_i, m_j\} + \{dm_j, m_i\} = d_{ijk} dm_k \quad (21)$$

and using (5) we invert this into

$$dm_k = \frac{N}{2(N-2)} d_{kij} (\{dm_i, m_j\} + \{dm_j, m_i\})$$

Consequently the symmetric combination (21) yields the $SO(N-1)$ vector one-form

$$X^i = X_\mu^i dx^\mu = \partial_\mu m_i^a T^a dx^\mu = \frac{N}{2(N-2)} d_{ijk} (\{dm_j, m_k\} + \{dm_k, m_j\}) \quad (22)$$

The antisymmetric combination yields a $SO(N-1)$ antisymmetric tensor one-form

$$Y^{ij} = Y_\mu^{ij} dx^\mu = -Y_\mu^{ji} dx^\mu = \{dm_i, m_j\} - \{dm_j, m_i\} \quad (23)$$

$$= d^{abc} (\partial_\mu m_i^a m_j^b - \partial_\mu m_j^a m_i^b) T^c dx^\mu \quad (24)$$

Finally, we define the $SO(N-1)$ symmetric tensor one-form

$$Z^{ij} = Z_\mu^{ij} dx^\mu = Z_\mu^{ji} dx^\mu = [dm_i, m_j] = f^{abc} \partial_\mu m_i^a m_j^b T^c dx^\mu \quad (25)$$

Under $SO(N - 1)$ this decomposes into the traceless symmetric tensor representation and the trace *i.e.* singlet representation, but we use it as is.

Observe that that (22), (24) and (25) are the *only* invariant one-forms that can be constructed using the variables m_i and natural $SU(N)$ invariant concepts. In particular, X^i , Y^{ij} and Z^{ij} are orthogonal to m^i , hence they determine a basis in the corresponding subspace $SU(N)/U(1)^{N-1}$. We can identify them as a basis of roots in $SU(N)$.

The dimension of the $SO(N - 1)$ vector representation (22) is $N - 1$. The dimension of the antisymmetric tensor representation (24) is $\frac{1}{2}(N - 1)(N - 2)$. The sum of these coincides with the dimension of (25),

$$N - 1 + \frac{1}{2}(N - 1)(N - 2) = \frac{1}{2}N(N - 1)$$

Moreover, we find that the $U(1)$ generators (19) map X_μ^i and Y_μ^{ij} into Z_μ^{ij} and *vice versa*,

$$\delta^i X^j = Z^{ij} \tag{26}$$

$$\delta^i Y^{jk} = d^{ikl} Z^{jl} - d^{ijl} Z^{kl} \tag{27}$$

and

$$\begin{aligned} \delta^i Z^{jk} &= -\frac{1}{N}(\delta_{ij} X^k + \delta_{ik} X^j) \\ &+ \frac{1}{4}(d_{jkl} d_{lin} - d_{jil} d_{lkn} - d_{kil} d_{ljn}) X^n + \frac{1}{4}(d_{jkl} Y^{il} + d_{jil} Y^{kl} + d_{kil} Y^{jl}) \end{aligned} \tag{28}$$

We note that this determines a natural complex structure.

We have now constructed four different sets of $SU(N)$ Lie-algebra valued forms (in μ) from the m_i . Each of these four sets induces an irreducible representation of $SO(N - 1)$, they decompose into the vector, the antisymmetric tensor, and the traceless symmetric tensor plus scalar representations of $SO(N - 1)$. The cotangent bundle to the co-adjoint orbit $SU(N)/U(1)^{N-1}$ is spanned by the one-forms (in μ) X^i , Y^{ij} and Z^{ij} .

By construction $(m_i, X^i, Y^{ij}, Z^{ij})$ yields a complete set of basis states for the $SU(N)$ Lie algebra, and can be used to decompose generic $SU(N)$ connections. For this we need appropriate dual variables that appear as coefficients. We first note that the connection A_μ^a is a $SU(N)$ Lie-algebra valued one-form, and the $SO(N - 1)$ acts on it trivially. Consequently the variable which is dual to m_i must be a one-form which transforms as a vector under $SO(N - 1)$. We call it C^i . The variables which are dual to the (X^i, Y^{ij}, Z^{ij}) are zero-forms, and in order to form invariant combinations they must decompose under the action of $SO(N - 1)$ in the same manner as X^i , Y^{ij} and Z^{ij} . Since we have also found a natural complex structure which is determined by the δ^i , this suggests that we denote these dual variables by $\phi^{ij} = \rho^{ij} + i\sigma^{ij}$. Here ρ^{ij} is dual to the Z^{ij} and can be decomposed into a traceless symmetric tensor and a singlet under $SO(N - 1)$. The σ^{ij} is

dual to the X^i and Y^{ij} . It decomposes into a vector and an antisymmetric tensor under $SO(N-1)$. The ($SO(N-1)$ invariant) $SU(N)$ connection A_μ^a then decomposes into

$$A = C \cdot m + (1 + \rho)[dm, m] + \sigma\{dm, m\} \quad (29)$$

Exactly in four dimensions this contains the correct number of independent variables for a general $SU(N)$ connection.

The ensuing curvature two form $F = dA + AA$ is obtained by a direct substitution. Of particular interest is the structure of F in the direction m_i , the maximal torus,

$$F_{\mu\nu}^a = m_i^a (\partial_\mu C_\nu^i - \partial_\nu C_\mu^i) - m_i^a \Omega_{\mu\nu}^i + \dots \quad (30)$$

Here the Ω^i are the Kirillov two forms (17), the terms that we have not presented explicitly depend on ϕ_{ij} and/or are in the direction of $SU(N)/U(1)^{N-1}$. (If we evaluate the curvature two form for the $SU(N)$ Cho connection (3), we find exactly the terms in (30).) Consequently F is a generating functional for the Kirillov two forms. In particular, for a flat connection we have

$$dC^i = \Omega^i$$

In [1], using Wilsonian renormalization group arguments we suggested that for $SU(2)$ (1) the following action

$$S = \int d^4x (\partial_\mu n)^2 + \frac{1}{e^2} (n, dn \wedge dn)^2 \quad (31)$$

may be relevant in the infrared limit of Yang-Mills theory. This is interesting since (31) describes knotlike configurations as solitons. The (self)linking of these knots is computed by the Hopf invariant

$$Q = \int F \wedge A \quad (32)$$

where

$$F = dA = (n, dn \wedge dn)$$

The present construction suggests a natural generalization of (31) to $SU(N)$,

$$S = \int d^4x (\partial_\mu m_i)^2 + \frac{1}{e_i^2} ([\partial_\mu m_i, \partial_\nu m_i])^2 \quad (33)$$

which reduces to (31) for $N = 2$. Since $\pi_3(SU(N)/U(1)^{N-1}) = \mathbb{Z}$ we expect that in the general case of $SU(N)$ we also have solitons. It would be interesting to understand their detailed structure.

As an example we consider $SU(3)$, the gauge group of strong interactions. There are two $SU(3)$ Lie-algebra valued vectors m_i^a which we denote by m^a and n^a respectively. We have $m^2 = n^2 = 1$ and $m^a n^a = 0$, and (2) becomes

$$A_\mu^a = B_\mu m^a + C_\mu n^a + f^{abc} \partial_\mu m^b m^c + f^{abc} \partial_\mu n^b n^c \quad (34)$$

$$+ \rho_{mm} f^{abc} \partial_\mu m^b m^c + \rho_{mn} f^{abc} \partial_\mu m^b n^c + \rho_{nn} f^{abc} \partial_\mu n^b n^c \quad (35)$$

$$+ \sigma_{mm} \partial_\mu m^a + \sigma_{nn} \partial_\mu n^a + \sigma_{mn} d^{abc} \partial_\mu m^b n^c \quad (36)$$

Here (34) is the $SU(3)$ Cho connection, and (35), (36) are the components of A_μ^a in the direction of the orbit $SU(3)/U(1) \times U(1)$. These components transform into each other under the action of the operators

$$\delta_m = [\bullet , m] \quad \delta_n = [\bullet , n]$$

In the Gell-Mann basis the vectors m and n satisfy

$$\begin{aligned} [m, n] &= 0 & \{m, m\} &= \frac{1}{\sqrt{3}}n \\ \{n, n\} &= -\frac{1}{\sqrt{3}}n & \{m, n\} &= \frac{1}{\sqrt{3}}m \end{aligned}$$

Notice that n is represented in terms of m , a unit vector with six independent field degrees of freedom to parametrize $SU(3)/U(1) \times U(1)$.

The action of δ_m and δ_n on $SU(3)/U(1) \times U(1)$ can be diagonalized. We find that δ_m corresponds to the adjoint action of the Cartan element λ_3 and δ_n to the adjoint action of the Cartan element λ_8 . This decomposes the $SU(3)/U(1) \times U(1)$ components of A_μ^a into $U(1)$ multiplets. In particular, $\delta_m^2 + \delta_n^2$ is a projection operator onto the basis one-forms of $SU(3)/U(1) \times U(1)$. Finally, the Kirillov symplectic two-forms are

$$\Omega_m = f^{abc} m^a (\partial_\mu m^b \partial_\nu m^c + \partial_\mu n^b \partial_\nu n^c)$$

and

$$\Omega_n = f^{abc} n^a (\partial_\mu m^b \partial_\nu m^c + \partial_\mu n^b \partial_\nu n^c)$$

and the action (33) is

$$S = \int d^4x \left((\partial_\mu m)^2 + (\partial_\mu n)^2 + \frac{1}{e_m^2} (f^{abc} \partial_\mu m^b \partial_\nu m^c)^2 + \frac{1}{e_n^2} (f^{abc} \partial_\mu n^b \partial_\nu n^c)^2 \right)$$

In conclusion, we have presented a group theoretical decomposition of four dimensional $SU(N)$ connection A_μ^a , which we argue is complete in the sense described in [1]. This decomposition involves a number of natural group theoretical concepts, and in particular relates the $SU(N)$ curvature two form to the Kirillov symplectic two forms on

the $SU(N)$ coadjoint orbits. Curiously, we also find that the components of A in the direction of these orbits can be decomposed according to irreducible representations of $SO(N - 1)$, with a natural complex structure. Our construction suggests a new class of nonlinear models generalizing the model first proposed in [6]. These models may have interesting properties, including the possibility of solitons with nontrivial topological structures [7].

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References

- [1] L. Faddeev and A. J. Niemi, hep-th/9807069
- [2] T.T. Wu and C.N. Yang, in *Properties of Matter Under Unusual Conditions*, edited by H. Mark and S. Fernbach (Interscience, New York 1969)
- [3] E. Witten, Phys. Rev. Lett. **38** (1977) 121
- [4] Y. M. Cho, Phys. Rev. **D21** (1980) 1080; *ibid.* **D23** (1981) 2415; Phys. Rev. Lett. **44** (1980) 1115
- [5] V. Periwal, hep-th/9808127
- [6] L. Faddeev, *Quantisation of solitons*, preprint IAS Print-75-QS70 ,1975; and in *Einstein and several contemporary tendencies in the field theory of elementary particles* in Relativity, quanta and cosmology vol. 1, M. Pantaleo and F. De Finis (eds.), Johnson Reprint, 1979
- [7] L. Faddeev and A. J. Niemi, Nature **387** (1997) 58