KORTEWEG-DE VRIES EQUATION: A COMPLETELY INTEGRABLE HAMILTONIAN SYSTEM

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The Korteweg-de Vries equation (KdV) arose long ago in an approximate theory of hydrodynamic waves,

$$u_t - 6uu_x + u_{xxx} = 0; \quad u(x, t)|_{t=0} = u(x); \quad -\infty < x < \infty; \; u(x) \to 0, \quad |x| \to \infty; \tag{1}$$

recently it has become the object of intensive study [1-3, 12]. A group of scholars, including Gardner, Green, Zabusky, Kruskal, and Miura, has made the following two important observations:

1. Equation (1) with smooth initial data admits an infinite set of first integrals. These integrals have local densities, i.e., they are representable in the form $I_n[u] = \int\limits_{-\infty}^{\infty} P_n(u, u_x, \ldots)\, dx$, where $P_n(u, u_x, \ldots)$ is a polynomial in u and spatial derivatives of u with orders up to n−2, which contains the term u^n. The first three such polynomials have the form $P_1(u) = u$, $P_2(u) = u^2$, $P_3(u, u_x) = u^3 + (u_x^2/2)$. An explicit form for eleven of the P_n is given in [2, 3]; in [3] an explicit procedure for determining them is given. An alternate approach for determining the $P_n(u, u_x, \ldots)$ has been developed by Lax [4].

2. An explicit solution of the KdV equation can be obtained by using the formalism of a scattering problem for the Schroedinger equation,

$$-\psi_{xx} + u(x)\psi = k^2\psi. \tag{2}$$

We clarify this in more detail. If

$$\int\limits_{-\infty}^{\infty} (1 + |x|)\, |u(x)|\, dx < \infty, \tag{3}$$

then Eq. (2) has a two-fold positive continuous spectrum and a finite number of negative characteristic values $-\varkappa_l^2$, $l = 1, \ldots, n$. For proof, see, for example, [5]. Let r(k) be the coefficient of reflection on the left, i.e., a function involving the solution $\psi(x, k)$ of Eq. (2) in the asymptotics for $x \to -\infty$, this function being uniquely defined by the conditions

$$\psi(x, k) = e^{ikx} + r(k) e^{-ikx} + o(1), \; x \to -\infty; \quad \psi(x, k) = t(k) e^{ikx} + o(1), \qquad x \to \infty. \tag{4}$$

Further let $\psi_l(x)$ be the characteristic functions of the discrete spectrum, normalized by the condition $\psi_l(x) = e^{\varkappa_l x}(1 + o(1))$, $x \to -\infty$, and c_l, $l = 1, \ldots, m$, the corresponding normalizing factors being $c_l = \left(\int\limits_{-\infty}^{\infty} \psi_l^2(x)\, dx\right)^{-1}$. The set $s = (r(k), \varkappa_l, c_l)$ will be called the scattering data for Eq. (2). The mapping $u(x) \to s$ of potentials u(x) into the scattering data s is uniquely invertible. The corresponding procedure for recovering u(x) from s, which is the inverse scattering problem, was formulated for the first time in [6] and investigated rigorously in [5]. In [5] necessary and sufficient conditions on the scattering data, corresponding to potentials satisfying condition (3), were obtained.

The remarkable result given in [1] consists in the following. In the set of scattering data we consider the action of the one-parameter group

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$$r(k) \to e^{8ik^3t} r(k), \quad \varkappa_l \to \varkappa_l, \quad c_l \to e^{8\varkappa_l^3 t} c_l. \tag{5}$$

It proves to be the case that the corresponding motion in the set of potentials $u(x) \to u(x, t)$ determines the solution $u(x, t)$ of the KdV equation.

In the present paper we give a new interpretation of this result and a new derivation based on it. This interpretation provides, in our opinion, a simple explanation of the somewhat puzzling conclusions given in [2].

Our interpretation may be formulated in the following way. The KdV equation is a completely integrable Hamiltonian system. The mapping $u \to s$ plays the role of a transformation, transforming the variables $u(x)$ into canonical variables of the type involving angle and action variables (see, for example [7]).

In order to justify these assertions we must:

1) producea simplicial form Ω on the set of potentials $u(x)$ and a Hamiltonian function $H[u]$ on this set, which generate the KdV equation according to the rules of Hamiltonian mechanics (see [8], for example);

2) calculate the preimages of the form Ω and the Hamiltonian $H[u]$ under the mapping $u \to s$ and express the canonical variables in the action-angle form in terms of the scattering data.

The first problem is easily solved. It is not hard to see that the KdV equation may be written in the form

$$u_t = \frac{d}{dx} \frac{\delta I_3[u]}{\delta u(x)}, \tag{6}$$

where the symbol $\delta H[u]/\delta u(x)$ denotes the gradient (Frechet derivative) of the function $H[u]$. It was pointed out in [4] that this result is due to Gardner.

The notation (6) for the KdV equation is clearly Hamiltonian. The corresponding simplicial form

$$\Omega(\delta_1 u, \delta_2 u) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dy \, [\delta_1 u(x) \delta_2 u(y) - \delta_1 u(y) \delta_2 u(x)] \tag{7}$$

has constant coefficients in the variables u and is therefore closed. We are using here the older but more natural and, for our infinite dimensional case, more suitable coordinate notation for a differential form in terms of the "local coordinates" $u(x)$ and their differentials, the variations $\delta_1 u(x)$ and $\delta_2 u(x)$. The role of the Hamiltonian $H[u]$ is played by the integral of motion

$$H[u] = I_3[u] = \int_{-\infty}^{\infty} \left(u^3(x) + \frac{1}{2} u_x^2\right) dx.$$

We devote the major part of this paper to a solution of the second problem. In §3 we express the Hamiltonian $H[u]$ in terms of the scattering data in the following way:

$$H[u] = -\frac{8}{\pi} \int_{-\infty}^{\infty} k^4 \ln(1 - |r(k)|^2) dk - \frac{32}{5} \sum_{l=1}^{m} \varkappa_l^5. \tag{8}$$

We will show that this expression is a special case of the formulas for traces [9, 10, 11]. Simultaneously we obtain explicit formulas for all the first integrals $I_n[u]$, deriving thereby simple recursion relations for the densities $P_n(u, u_x, \ldots)$. In terms of the scattering data these integrals may be expressed by formulas analogous to Eq. (8); i.e., they involve moments of the function $\ln(1 - |r(k)|^2)$ and powers of $\varkappa_l$.

In §2, using the formalism of the inverse scattering problem, we express the form Ω in terms of the scattering data and we find the corresponding canonical variables. We show, in particular, that $P(k) = -(k/\pi) \ln(1 - |r(k)|^2)$, $p_l = \varkappa_l^2$, $l = 1, \ldots, m$, are variables of impulse type, so that the fact of the constancy of the integrals $I_n[u]$ becomes trivial. Solution of the Hamiltonian equations may thus be trivialized with the corresponding answer supplied by the formulas (5).

In §1, as a preliminary, we supply, without going into a detailed derivation, the necessary facts of scattering theory for the one-dimensional Schroedinger equation over the entire axis in a form suitable to our purposes.