How Algebraic Bethe Ansatz works for integrable model

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1 Introduction

In my Les–Houches lectures of 1982 I described the inverse scattering method of solving the integrable field–theoretical models in 1+1 dimensional space–time. Both classical case, stemming from the famous paper by Gardner, Green, Kruskal and Miura of 1967 on KdV equation, and its quantum counterpart, developed mostly by Leningrad group around 1978–79, were discussed. In particular, the algebraic way of deriving the Bethe–Ansatz equations was presented, but its use was not illustrated in any detail. I just stated in the end of the lectures, that "The hard work just begins".

In this course I shall exactly describe this work. During last 10 years I lectured several times on this subject, but this particular lecture course is the longest and more detailed.

In the announcement of the School my course is called "Hidden symmetries in integrable models". The term "symmetry" in modern literature on mathematical physics is supplied with several adjectives, such as hidden, dynamical, broken, deformed etc., but there is no exact definition for all this. Thus I decided to change my title in the proceedings to reflect more adequately the actual content of the course.

The term "integrable models" in the title refers to particular family of quantum field–theoretical models in 1+1 dimensional space–time, which are solvable by means of the quantum variant of inverse scattering method. The adjective "integrable" stems from a paper of Zakharov and me of 1971 where the KdV equation was shown to allow the interpretation as an integrable (though infinite–dimensional) hamiltonian system.

The most famous example of the integrable model is the Sine–Gordon equation

$$\Box\varphi + m^2 \sin\varphi = 0 \tag{1}$$

for the scalar field $\varphi(x,t)$, which is relativistic and nonlinear.

As is well known, the quantum field–theoretic models with interaction are plagued by infinities and some regularization is necessary. The discretizing of the space, or going to the lattice, is one way for such a regularization. It reduces the field model in the finite volume to a system with finite number of degrees of freedom.

In the beginning of 80–ties, due mainly to the work of Izergin and Korepin it was realized, that the integrable models allow the lattice counterparts, which are also integrable and can be interpreted as the quantum spin models of magnetic chains. This unexpected but very welcome connection showed the universality of the spin chains in the domain of integrable models. In my course I shall begin with and speak in detail on the spin chains. The field–theoretical models will appear as their particular continuous space limits.

One can ask, what is good in $1+1$ models, when our space–time is $3+1$–dimensional. There are several particular answers to this question.

1. The toy models in $1+1$ dimension can teach us about the realistic field–theoretical models in a nonperturbative way. Indeed such phenomena as renormalization, asymptotic freedom, dimensional transmutation (i.e. the appearance of mass via the regularization parameters) hold in integrable models and can be described exactly.

2. There are numerous physical applications of the $1+1$ dimensional models in the condensed matter physics.

3. The formalism of integrable models showed several times to be useful in the modern string theory, in which the world sheet is 2–dimensional anyhow. In particular the conformal field theory models are special massless limits of integrable models.

4. The theory of integrable models teaches us about new phenomena, which were not appreciated in the previous developments of Quantum Field Theory, especially in connection with the mass spectrum.

5. I cannot help mentioning that working with the integrable models is a delightful pastime. They proved also to be very successful tool for the educational purposes.

These reasons were sufficient for me to continue to work in this domain for last 25 years (including 10 years of classical solitonic systems) and teach quite a few of followers, often referred to as Leningrad–St.Petersburg school.

I am very grateful to the organizers of the school Professors A.Connes and K.Gawedsky for inviting me to lecture. First, it is always nice to be in Les–Houches and it is already my third lecture course here. Second, the texts of two previous lectures were transformed into monographs later. I hope that this course will eventually lead to one more book dedicated to the quantum theory of solitons.

2 General outline of the course.

My usual methodological trick in teaching is not to begin in full generality but rather to choose a representative example and explain on it all technical features in such a way, that generalization become reasonably evident. Thus I begin the course with the concrete example of the magnetic model – the spin 1/2 XXX chain.

All the ingredients of Algebraic Bethe Ansatz, which is another name for the Quantum Inverse Scattering method, namely Lax operator, derivation of Bethe–Ansatz equations, thermodynamic limit, will be presented in full detail on this example. Then the generalization to XXX model of spin $s > 1/2$ and XXZ model will follow. Here only features, which distinguish this model from the basic one, will be described. Finally the continuous field–theoretical models such as Nonlinear Schroedinger Equation, S^2 nonlinear σ–model, principal chiral field model for $SU(2)$ and Sine–Gordon model will be included as particular limits of spin chains.

We will see, that in the description of dynamics the finite time shift

$$U = e^{-iH\Delta} \tag{2}$$

will appear naturally. This will make our discretization scheme more consistent, time and space being discrete simultaneously.

Now I shall present some kinematics of the models we shall consider.

As "space" we shall consider a discrete circle, namely the ordered set of points, labeled by integers n with the identification $n \equiv n + N$, where N is a fixed positive integer. As a "fundamental domain" we shall take $n = 1, \ldots, N$. The integer N plays the role of the volume of the space; the identification reflects the periodic boundary condition.

Formal continuous limit will be described by introducing the lattice spacing Δ and coordinate $x = n\Delta$ which becomes continuous in the limit $\Delta \to 0$, $N \to \infty$. In particular the following rule will be used

$$\frac{\delta_{mn}}{\Delta} = \delta(x - y) \quad , \tag{3}$$

so that the Kroneker symbol δ_{mn} is of order Δ.

The quantum algebra of observables $\mathcal{A}$ is generated by dynamical variables X_n^α, attached to each lattice site n. Index α assumes some finite number of values. The algebra $\mathcal{A}$ is defined by fixing the set of commutation relations between X_n^α. These relations are called ultralocal, when X_m^α and X_n^β commute for $n \neq m$. A more relaxed condition of locality is that X_m^α and X_n^β do not commute only for $|n - m|$ small, in particular for $n = m - 1$ and $n = m + 1$.

Let us present examples, beginning with ultralocal case.

1. Canonical variables $\varphi_n^\alpha, \pi_n^\alpha, \alpha = 1, \ldots, l$ with the relations

$$[\varphi_n^\alpha, \varphi_m^\beta] = 0 \ , \quad [\pi_n^\alpha, \pi_m^\beta] = 0 \ , \tag{4}$$

$$[\varphi_m^\alpha, \pi_n^\beta] = i\hbar \mathrm{I} \delta_{nm} \delta_{\alpha\beta} \ . \tag{5}$$

Here [,] is used for commutator

$$[a, b] = ab - ba \quad , \tag{6}$$

$\hbar$ is a Planck constant, which we soon shall drop, I is as unity in algebra $\mathcal{A}$. We can call l a number of degrees of freedom per lattice site; the full number of degrees of freedom is Nl.

Each pair of canonical variables is represented in an infinite dimensional Hilbert space, which can be chosen as $L_2(\mathrm{R})$ with φ and π being operators of multiplication and differentiation

$$\varphi \ f(\xi) = \xi f(\xi) \ ; \quad \pi \ f(\xi) = \frac{\hbar}{i} \frac{d}{d\xi} f(\xi) \quad . \tag{7}$$

2. Spin variables S_n^α, $\alpha = 1, 2, 3$ with the relations

$$[S_m^\alpha, S_n^\beta] = i\mathrm{I}\hbar \varepsilon_{\alpha\beta\gamma} S_n^\gamma \delta_{mn} \quad , \tag{8}$$

where $\varepsilon_{\alpha\beta\gamma}$ is a completely antisymmetric tensor, $\varepsilon_{123} = 1$.

Mathematically these variables define a Lie algebra $sl(2)$, the finite dimensional representations of which are labeled by half–integer $s = 0,\ 1/2,\ 1,\ \ldots$ and are realized in C^{2s+1}. In the smallest nontrivial dimension ($s = 1/2$) operators S_n^α are represented by Pauli matrices σ^α

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

as $S_n^\alpha = \hbar/2\sigma^\alpha$.

3. Weyl variables.
   For one degree of freedom per lattice site these variables consist of a pair u_n, v_n with the exchange relations

$$u_m u_n = u_n u_m \; ; \quad v_m v_n = v_n v_m \; ; \quad u_m v_n = v_n u_m \; , \quad m \neq n \; ;$$
$$u_n v_n = q v_n u_n \; , \tag{10}$$

   where q is a given complex number. Usually it assumes the values on a circle and is parametrized by a real number γ

$$q = e^{i\hbar\gamma} \quad . \tag{11}$$

4. q–deformed spin variables S_n^α. In writing the commutation relations we shall drop the lattice index n and present them for fixed n as follows

$$q^{S^3} S^{\pm} = q^{\pm 1} S^{\pm} q^{S^3} \quad ; \tag{12}$$

$$[S^+, S^-] = \frac{(q^{S^3})^2 - (q^{S^3})^{-2}}{q - q^{-1}} \qquad , \tag{13}$$

   via generators denoted by q^{S^3}, S^+ and S^-. When $q \to 1$ (or $\gamma \to 0$) these relations turn into the usual $sl(2)$ relations for S^α, $S^\pm$ being the usual combinations $S^\pm = S^1 \pm i S^2$.

   Thus they define a q–deformed $sl(2)$ algebra, denoted by $sl_q(2)$. For generic q the finite dimensional representations are given in the same spaces C^{2s+1} as in nondeformed case. However for q on the circle some new interesting representations occur, more on this below.

The Hilbert space for the representation of ultralocal algebra $\mathcal{A}$ has a natural tensor–product form

$$\mathcal{H} = \prod_{n=1}^{N} \otimes h_n = h_1 \otimes h_2 \otimes \ldots \otimes h_n \otimes \ldots \otimes h_N \tag{14}$$

(where all h_n could be the same) and variables X_n^α act nontrivially only in the space h_n

$$X_n^\alpha = \mathrm{I} \otimes \mathrm{I} \otimes \ldots \underbrace{\otimes X^\alpha \otimes}_{n\text{–th place}} \ldots \otimes \mathrm{I} \; . \tag{15}$$

In the continuous limit we encounter the problem of considering the infinite tensor products, which is known to be intricated from the time of v.Neumann. In fact, the concrete examples of the continuous limits will give the instances of the rather nontrivial constructions of such products.

The simplest example of the nonultralocal relation is furnished by the exchange algebra for variables w_n with the relations

$$w_n w_{n+1} = q w_{n+1} w_n \, ; \quad w_m w_n = w_n w_m \quad |n - m| \geq 2 \; . \tag{16}$$

The Hilbert space for this algebra is also a tensor product, but with the length being around $N/2$. We shall not discuss it here, however, the example being given just for illustration.

General considerations *a-lá* Darboux theorem in classical mechanics state that the canonical variables are generic in the sense that all other types of dynamical variables can be expressed through them. Let us illustrate it on the examples above.

Let ψ, ψ^* be a complex pair of canonical variables with relation

$$[\psi, \psi^*] = \mathrm{I} \tag{17}$$

(i.e. $\psi = \frac{1}{\sqrt{2}}(\varphi + i\pi)$, $\psi^* = \frac{1}{\sqrt{2}}(\varphi - i\pi)$); then the variables

$$S^+ = S^1 + iS^2 = \psi^*(2s - \psi^*\psi) \ ; \tag{18}$$

$$S^- = S^1 - iS^2 = \psi \ ; \tag{19}$$

$$S^3 = \psi^*\psi - s \tag{20}$$

satisfy the spin commutation relation. Here s is any complex parameter, which can be called spin, as the Casimir $C = (S^1)^2 + (S^2)^2 + (S^3)^2$ has value $C = s(s+1)$.

Weyl pair u, v can be realized via canonical variables π and φ as follows

$$u = e^{i\pi Q} \ , \quad v = e^{i\varphi P} \ , \tag{21}$$

where P and Q are any complex numbers and $q = e^{iPQ}$.

Finally, the deformed spin variables q^{S^3}, $S^\pm$ can be realized as

$$S^\pm = e^{\pm i\pi/2}(1 + m^2 e^{\pm 2i\gamma\varphi})e^{\pm i\pi/2} \ ; \tag{22}$$

$$q^{S^3} = e^{i\gamma\varphi} \ , \tag{23}$$

where m^2 and γ are parameters.

Of course the Hilbert space for the canonical variables is infinite dimensional and the representations for the spin variables appear as reductions. For example, if $2s$ is integer then the subspace consisting of $2s+1$ states ω, $\psi^*\omega$, $\dots (\psi^*)^{2s}\omega$, where ω is a vector annihilated by ψ, $\psi\omega = 0$, is invariant with respect to action of $S^\pm$, S^3.

The Weyl pair u, v in form (21) has finite dimensional representation if q is a root of unity, or when $PQ/2\pi$ is a rational number. Corresponding reduction exists also for the deformed spin variables if $\gamma/2\pi$ is rational and it gives the so-called finite dimensional cyclic representation of $sl_q(2)$.

These reductions evidently reflect some global aspects of the would–to–be quantum Darboux theorem.

This completes my short introduction to the kinematics of the models I plan to consider. I come to the representative example — spin 1/2 XXX chain.

3 $\mathrm{XXX}_{1/2}$ model. Description.

The origin of the abbreviation XXX will become clear soon. As I have already said this is quantum model, defined in a Hilbert space

$$\mathcal{H}_N = \prod_{n=1}^{N} \otimes h_n \ , \tag{24}$$

where each local space h_n is two–dimensional

$$h_n = \mathrm{C}^2 \quad . \tag{25}$$

The spin variables S_n^α are acting on each h_n as Pauli matrices divided by 2.

There are several important observables, such as the total spin

$$S^\alpha = \sum_n S_n^\alpha \tag{26}$$

or the total hamiltonian

$$H = \sum_{\alpha,n} \left(S_n^\alpha S_{n+1}^\alpha - \frac{1}{4} \right) \quad , \tag{27}$$

where the periodicity

$$S_{n+N}^\alpha = S_n^\alpha \tag{28}$$

is to be taken into account. We have

$$[H, S^\alpha] = 0 \quad , \tag{29}$$

which reflects the $sl(2)$ symmetry of the model.

The abbreviation XXX is used to stress this invariance which is reflected in the fact that all coefficients in front of combination $S_n^\alpha S_{n+1}^\alpha$ in hamiltonian are equal. A more general hamiltonian

$$H = \sum_{\alpha,n} J^\alpha S_n^\alpha S_{n+1}^\alpha \tag{30}$$

with parameters J^α corresponds to XYZ spin 1/2 model.

Our problem is to investigate the spectrum of H. Of course for finite N it is just a problem about the matrix $2^N \times 2^N$ accessible by computer. However we will be interested in the limit $N \to \infty$, when only analytic methods work.

The core of our approach is a generating object, called Lax operator. This object is a rather long shot from the Lax operator of KdV equation but historically this name was fixed for any linear operator, entering the auxiliary spectral problem of the classical inverse scattering method.

The definition of the Lax operator involves the local quantum space h_n and the auxiliary space V, which for the beginning will be also C^2. Lax operator $L_{n,a}(\lambda)$ acts in $h_n \otimes V$ and is given explicitly by the expression

$$L_{n,a}(\lambda) = \lambda \mathrm{I}_n \otimes \mathrm{I}_a + i \sum_\alpha S_n^\alpha \otimes \sigma^\alpha \quad , \tag{31}$$

where I_n, S_n^α act in h_n and I_a, σ^α are unit and Pauli matrices in $V = \mathrm{C}^2$; λ is a complex parameter, usually called the spectral parameter, reminding its role as an eigenvalue in the original Lax operator.

Alternatively $L_{n,a}(\lambda)$ can be written as 2×2 matrix

$$L_{n,a}(\lambda) = \begin{pmatrix} \lambda + iS_n^3 & iS_n^- \\ iS_n^+ & \lambda - iS_n^3 \end{pmatrix} \quad , \tag{32}$$

acting in V with entries being operators in quantum space h_n. One more form uses the fact that operator

$$\mathrm{P} = \frac{1}{2}(\mathrm{I}\otimes\mathrm{I} + \sum_\alpha \sigma^\alpha \otimes \sigma^\alpha) \tag{33}$$

is a permutation in $\mathrm{C}^2 \otimes \mathrm{C}^2$

$$\mathrm{P}\; a\otimes b = b\otimes a \quad . \tag{34}$$

In terms of $\mathrm{P}_{n,a}$, which makes sense due to the fact that h_n and V are the same C^2, we have

$$L_{n,a}(\lambda) = (\lambda - \frac{i}{2})\mathrm{I}_{n,a} + i\mathrm{P}_{n,a} \quad . \tag{35}$$

Now we establish the main property of Lax operator — the commutation relation for its entries. As we have four of them we are to write down 16 relations. Our convenient notations allow to write them all in one line.

Consider two exemplars of Lax operators $L_{n,a_1}(\lambda)$ and $L_{n,a_2}(\mu)$ with the same quantum space and V_1 and V_2 serving as corresponding auxiliar spaces. The products $L_{n,a_1}(\lambda)\, L_{n,a_2}(\mu)$ and $L_{n,a_2}(\mu)\, L_{n,a_1}(\lambda)$ make sense in a triple tensor product $h_n \otimes V_1 \otimes V_2$. We claim, that these two products are similar operators with the intertwiner acting only in $V_1 \otimes V_2$ and so not containing quantum operators. In other words, there exists an operator $R_{a_1,a_2}(\lambda-\mu)$ in $V_1\otimes V_2$ such that the following relation is true

$$R_{a_1,a_2}(\lambda-\mu)\, L_{n,a_1}(\lambda)\, L_{n,a_2}(\mu) \;=\; L_{n,a_2}(\mu)\, L_{n,a_1}(\lambda)\, R_{a_1,a_2}(\lambda-\mu)\,. \tag{36}$$

The explicit expression for $R_{a_1,a_2}(\lambda)$ is

$$R_{a_1,a_2}(\lambda) = \lambda \mathrm{I}_{a_1,a_2} + i\mathrm{P}_{a_1,a_2} \quad , \tag{37}$$

where I_{a_1,a_2} and P_{a_1,a_2} are unity and permutation in $V_1 \otimes V_2$.

Comparing (35) and (37) we see that the Lax operator $L_{n,a}(\lambda)$ and operator $R_{a_1,a_2}(\lambda)$, which we shall call R–matrix, are essentially the same.

To check (36) it is convenient to use the form (37) for $L_{n,a}(\lambda)$ and the commutation relation for permutations

$$\mathrm{P}_{n,a_1}\,\mathrm{P}_{n,a_2} \;=\; \mathrm{P}_{a_1,a_2}\,\mathrm{P}_{n,a_1} \;=\; \mathrm{P}_{n,a_2}\,\mathrm{P}_{a_2,a_1} \tag{38}$$

together with evident symmetry

$$\mathrm{P}_{a_2,a_1} \;=\; \mathrm{P}_{a_1,a_2} \quad . \tag{39}$$

The importance of the relation (36) will become clear momentarily. We shall call it in what follows the fundamental commutation relation (FCR). Its general place in the family of the Yang–Baxter relations will become evident later.

The Lax operator $L_{n,a}(\lambda)$ has a natural geometric interpretation as a connection along our chain, defining the transport between sites n and $n{+}1$ via the Lax equation

$$\psi_{n+1} \;=\; L_n \psi_n \tag{40}$$

for vector $\psi_n \;=\; \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix}$ with entries in $\mathcal{H}$. The ordered product over all sites between n_2 and n_1

$$T_{n_1,a}^{n_2}(\lambda) = L_{n_2,a}(\lambda)\,\ldots\, L_{n_1,a}(\lambda) \tag{41}$$

defines the transport form n_1 to n_2+1 and the full product

$$T_{N,a}(\lambda) \;=\; L_{N,a}(\lambda)\,\ldots\,L_{1,a}(\lambda) \tag{42}$$

is a monodromy around our circle. The last operator is given as a 2×2 matrix in the auxiliary space V

$$T_{N,a} = \begin{pmatrix} A_N(\lambda) & , & B_N(\lambda) \\ C_N(\lambda) & , & D_N(\lambda) \end{pmatrix} \tag{43}$$

with entries being operators in the full quantum space $\mathcal{H}$. As in classical case the map from local dynamical variables (S_n^α) to monodromy $T_{N,a}(\lambda)$ is a tool for solving the dynamical problem. We shall see that $T_{N,a}(\lambda)$ is a generating object for main observables such as spin and hamiltonian, as well as for the spectrum rising operators.

For that we shall establish first the FCR for $T_{N,a}(\lambda)$. We claim, that it has exactly the same form as the local FCR (36), namely

$$R_{a_1,a_2}(\lambda-\mu)\,T_{a_1}(\lambda)\,T_{a_2}(\mu) \;=\; T_{a_2}(\mu)\,T_{a_1}(\lambda)\,R_{a_1,a_2}(\lambda-\mu)\;. \tag{44}$$

We dropped here the index N and shall do it in what follows as soon as it does not lead to confusion.

Derivation of (44) is very simple and uses the advantages of our notations. We shall prove it for any transport operator. It is clear, that it is enough to consider the transport along two sites, i.e. $n\to n+2$. With short notations $R_{a_1,a_2}(\lambda-\mu)=R_{12}$, $L_{n,a_1}(\lambda)=L_1$, $L_{n+1,a_1}(\lambda)=L_1'$, $L_{n,a_2}(\mu)=L_2$, $L_{n+1,a_2}(\mu)=L_2'$ we have

$$\begin{aligned} R_{12}L_1'L_1L_2'L_2 &= \text{(due to commutativity of } L_1 \text{ and } L_2')\\ =R_{12}L_1'L_2'L_1L_2 &= \text{(due to the local FCR for } L_1,\, L_2 \text{ and } L_1',\, L_2')\\ =L_2'L_1'L_2L_1R_{12} &= \text{(due to commutativity of } L_1' \text{ and } L_2)\\ =L_2'L_2L_1'L_1R_{12} &\quad . \end{aligned}$$

This completes the proof.

The monodromy $T_{N,a}(\lambda)$ is a polynomial in λ of order N

$$T_{N,a}(\lambda) = \lambda^N + i\lambda^{N-1}\sum_\alpha (S^\alpha\otimes\sigma^\alpha) + \ldots\quad , \tag{45}$$

so that total spin S^α appears via the coefficient of next to the highest degree. Now we shall find the place for the hamiltonian. The FCR (44) shows, that the family of operators

$$F(\lambda) = \operatorname{tr} T(\lambda) = A(\lambda) + D(\lambda) \tag{46}$$

is commuting

$$[F(\lambda), F(\mu)] = 0\quad . \tag{47}$$

Its nontrivial λ expansion begins with power λ^{N-2}

$$F(\lambda) = 2\lambda^N + \sum_{l=0}^{N-2} Q_l\lambda^l \tag{48}$$

and produces $N-1$ commuting operators Q_l. We shall show, that H belongs to this family.

For this note, that the point $\lambda = i/2$ is rather special

$$L_{n,a}(i/2) = i\mathrm{P}_{n,a} \tag{49}$$

and of course for any λ

$$\frac{d}{d\lambda} L_{n,a}(\lambda) = \mathrm{I}_{n,a} \quad . \tag{50}$$

This makes it easy to control the expansion of $F(\lambda)$ in the vicinity of $\lambda = i/2$.

We have

$$T_{N,a}(i/2) = i^N \mathrm{P}_{N,a}\, \mathrm{P}_{N-1,a} \ldots \mathrm{P}_{1,a} \quad . \tag{51}$$

This string of permutations is easily transformed into

$$\mathrm{P}_{1,2}\, \mathrm{P}_{2,3} \ldots \ \mathrm{P}_{N-1,N}\, \mathrm{P}_{N,a} \tag{52}$$

by taking permutations one after another from left to right and taking into account the properties (38) and (39) and commutativity of permutations with completely different indeces. Now the trace over the auxiliary space is easily taken

$$\mathrm{tr}_a \mathrm{P}_{N,a} \, = \, \mathrm{I}_N \quad , \tag{53}$$

so that

$$U = i^{-N} \mathrm{tr}_a T_N(i/2) = \mathrm{P}_{1,2}\, \mathrm{P}_{2,3} \ldots \ \mathrm{P}_{N-1,N} \quad . \tag{54}$$

It is easy to see, that U is a shift operator in $\mathcal{H}$. The property (34) can be rewritten as

$$\mathrm{P}_{n_1,n_2}\, X_{n_2}\, \mathrm{P}_{n_1,n_2} \, = \, X_{n_1} \quad . \tag{55}$$

Thus

$$\begin{aligned} X_n U = \mathrm{P}_{1,2} \ldots X_n\, \mathrm{P}_{n-1,n}\, \mathrm{P}_{n,n+1} \ldots \mathrm{P}_{N-1,N} \, = \\ \mathrm{P}_{1,2} \ldots \mathrm{P}_{n-1,n}\, X_{n-1}\, \mathrm{P}_{n,n+1} \ldots \mathrm{P}_{N-1,N} = U X_{n-1} \quad . \end{aligned} \tag{56}$$

Operator U is unitary

$$U^*\, U \, = \, U\, U^* \, = \, \mathrm{I} \quad , \tag{57}$$

because the permutations have properties

$$\mathrm{P}^* \, = \, \mathrm{P}\, ; \ \mathrm{P}^2 \, = \, \mathrm{I} \quad , \tag{58}$$

and we have

$$U^{-1}\, X_n\, U \, = \, X_{n-1} \quad , \tag{59}$$

which allows to introduce important observable — momentum. By definition, momentum P produces an infinitesimal shift and on the lattice it is substituted by shift along one site

$$e^{iP} \, = \, U \quad . \tag{60}$$

We proceed now to expand $F(\lambda)$ in the vicinity of $\lambda = i/2$. We get

$$\left. \frac{d}{d\lambda} T_a(\lambda) \right|_{\lambda = i/2} \, = \, i^{N-1} \sum_n \mathrm{P}_{N,a} \ldots \widehat{\mathrm{P}}_{n,a} \ldots \mathrm{P}_{1,a} \ , \tag{61}$$

where $\hat{}$ means that corresponding factor is absent. Repeating our trick we transform this after taking the trace over V into

$$\frac{d}{d\lambda} F_a(\lambda)\Big|_{\lambda=i/2} = i^{N-1} \sum_n \mathrm{P}_{1,2} \dots \mathrm{P}_{n-1,n+1} \dots \mathrm{P}_{N-1,N} \ . \tag{62}$$

We can cancel most of permutations here, multiplying by U^{-1}; as a result we get

$$\frac{d}{d\lambda} F_a(\lambda)\ F_a(\lambda)^{-1}\Big|_{\lambda=i/2} = \frac{d}{d\lambda} \ln F_a(\lambda)\Big|_{\lambda=i/2} = \frac{1}{i} \sum_n \mathrm{P}_{n,n+1} \ . \tag{63}$$

Using (33) we can rewrite the expression (27) for the hamiltonian as

$$H = \frac{1}{2} \sum_n \mathrm{P}_{n,n+1} - \frac{N}{2} \tag{64}$$

and comparing (64) and (63) we have

$$H = \frac{i}{2} \frac{d}{d\lambda} \ln F(\lambda)\Big|_{\lambda=i/2} - \frac{N}{2} \ . \tag{65}$$

Thus I have shown, that H indeed belongs to the family of $N-1$ commuting operators generated by the trace of monodromy $F(\lambda)$. One component of spin, say S^3, completes this family to N commuting operators. This is a proof of the integrability of the classical counterpart of our model, which can be considered as a system with N degrees of freedom. In the next section I shall show, how to describe the set of eigenstates of commuting family $F(\lambda)$ using the offdiagonal elements of monodromy.

4 $\mathbf{XXX}_{1/2}$ model. Bethe Ansatz equations.

Here I shall describe a procedure to diagonalize the whole family of operators $F(\lambda)$ in a rather algebraic fashion, based on the global FCR (44) and some simple properties of local Lax operators. In a way the working generalizes the simplest quantum mechanical treatment of harmonic oscillator hamiltonian $n = \psi^* \psi$ based on commutation relations $[\psi, \psi^*] = \mathrm{I}$ and existence of a state ω such that $\psi\,\omega = 0$.

Let us write the relevant set of FCR

$$[B(\lambda), B(\mu)] = 0 \ ; \tag{66}$$

$$A(\lambda)\, B(\mu) = f(\lambda-\mu)\, B(\mu)\, A(\lambda) + g(\lambda-\mu)\, B(\lambda)\, A(\mu) \ ; \tag{67}$$

$$D(\lambda)\, B(\mu) = h(\lambda-\mu)\, B(\mu)\, D(\lambda) + k(\lambda-\mu)\, B(\lambda)\, D(\mu) \ , \tag{68}$$

where

$$f(\lambda) = \frac{\lambda - i}{\lambda} \ ; \qquad g(\lambda) = \frac{i}{\lambda} \ ;$$

$$h(\lambda) = \frac{\lambda + i}{\lambda} \ ; \qquad k(\lambda) = -\frac{i}{\lambda} \ . \tag{69}$$

To read these relations from one line FCR (44) one must use the explicit matrix representation of FCR in $V \otimes V$. All objects in it are 4×4 matrices in a natural basis

$$e_1 = e_+ \otimes e_+ \,, \; e_2 = e_+ \otimes e_- \,, \; e_3 = e_- \otimes e_+ \,, \; e_4 = e_- \otimes e_- \tag{70}$$

in $C^2 \otimes C^2$, where

$$e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{71}$$

Matrix P assumes the form

$$\mathrm{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{72}$$

so that $R(\lambda)$ looks like

$$R(\lambda) = \begin{pmatrix} a(\lambda) & & & \\ & b(\lambda) & c(\lambda) & \\ & c(\lambda) & b(\lambda) & \\ & & & a(\lambda) \end{pmatrix} \tag{73}$$

(we do not write in the zeros), where

$$a = \lambda + i \,, \;\; b = \lambda \,, \;\; c = i \;\; . \tag{74}$$

The matrices $T_{a_1}(\lambda)$ and $T_{a_2}(\mu)$ take the form

$$T_{a_1}(\lambda) = \begin{pmatrix} A(\lambda) & & B(\lambda) & \\ & A(\lambda) & & B(\lambda) \\ C(\lambda) & & D(\lambda) & \\ & C(\lambda) & & D(\lambda) \end{pmatrix} \tag{75}$$

and

$$T_{a_2}(\mu) = \begin{pmatrix} A(\mu) & B(\mu) & & \\ C(\mu) & D(\mu) & & \\ & & A(\mu) & B(\mu) \\ & & C(\mu) & D(\mu) \end{pmatrix} . \tag{76}$$

Thus we have

$$T_{a_1}(\lambda)\, T_{a_2}(\mu) = \begin{pmatrix} A(\lambda)A(\mu) & A(\lambda)B(\mu) & B(\lambda)A(\mu) & B(\lambda)B(\mu) \\ A(\lambda)C(\mu) & A(\lambda)D(\mu) & B(\lambda)C(\mu) & B(\lambda)D(\mu) \\ C(\lambda)A(\mu) & C(\lambda)B(\mu) & D(\lambda)A(\mu) & D(\lambda)B(\mu) \\ C(\lambda)C(\mu) & C(\lambda)D(\mu) & D(\lambda)C(\mu) & D(\lambda)D(\mu) \end{pmatrix} \tag{77}$$

and $T_{a_2}(\mu)\, T_{a_1}(\lambda)$ is given by matrix, where in all matrix elements the factors have opposite order. Now to get (67) one is to use the $(1, 3)$ relation in FCR (44)

$$a(\lambda - \mu)\, B(\lambda)\, A(\mu) \;=\; c(\lambda - \mu)\, B(\mu)\, A(\lambda) \;+\; b(\lambda - \mu)\, A(\mu)\, B(\lambda) \tag{78}$$

and interchange $\lambda \leftrightarrow \mu$ to get

$$A(\lambda)\, B(\mu) \,=\, \frac{a(\mu-\lambda)}{b(\mu-\lambda)}\, B(\mu)\, A(\lambda) \,-\, \frac{c(\mu-\lambda)}{b(\mu-\lambda)}\, B(\lambda)\, A(\mu)\,. \tag{79}$$

Other relations are obtained similarly.

The exchange relations (67) and (68) substitute the relations

$$\psi\, n \,=\, (n+1)\psi\,, \quad \psi^*\, n \,=\, (n-1)\, \psi^* \tag{80}$$

for the harmonic oscillator with $n \,=\, \psi^*\psi$. Now the analogue of "highest weight" ω such as $\psi\,\omega \,=\, 0$ will be played by a reference state Ω such, that

$$C(\lambda)\, \Omega \,=\, 0 \quad . \tag{81}$$

To find this state we observe that in each h_n there exists a vector ω_n such that the Lax operator $L_{n,a}(\lambda)$ becomes triangular in the auxiliary space, when applied to it

$$L_n(\lambda)\, \omega_n \,=\, \begin{pmatrix} \lambda + \frac{i}{2} & * \\ 0 & \lambda - \frac{i}{2} \end{pmatrix} \omega_n \quad . \tag{82}$$

This vector is given by $\omega_n = e_+$. By $*$ we denote operator expressions which are not relevant for us. Now for vector Ω in $\mathcal{H}$

$$\Omega \,=\, \prod_n \otimes\, \omega_n \tag{83}$$

we get

$$T(\lambda)\, \Omega \,=\, \begin{pmatrix} \alpha^N(\lambda) & * \\ 0 & \delta^N(\lambda) \end{pmatrix} \Omega \;, \tag{84}$$

where

$$\alpha(\lambda) \,=\, \lambda + \frac{i}{2}\,, \quad \delta(\lambda) \,=\, \lambda - \frac{i}{2}\,. \tag{85}$$

In other words we have

$$C(\lambda)\, \Omega \,=\, 0\,; \;\; A(\lambda)\, \Omega \,=\, \alpha^N(\lambda)\, \Omega\,; \;\; D(\lambda)\, \Omega \,=\, \delta^N(\lambda)\, \Omega\,, \tag{86}$$

so that Ω is an eigenstate of $A(\lambda)$ and $D(\lambda)$ simultaneously and also that for $F \,=\, A \,+\, D$.

Other eigenvectors will be looked for in the form

$$\Phi(\{\lambda\}) \,=\, B(\lambda_1)\, \dots\, B(\lambda_l)\, \Omega\,. \tag{87}$$

The condition that $\Phi(\{\lambda\})$ is an eigenvector of $F(\lambda)$ will lead to a set of algebraic relations on parameters $\lambda_1, \dots \lambda_l$. I proceed to derive these equations.

Using the exchange relations (67) we get

$$A(\lambda)\, B(\lambda_1)\, \dots\, B(\lambda_l)\, \Omega \,=\, \prod_{k=1}^{l} f(\lambda-\lambda_k)\, \alpha^N(\lambda)\, B(\lambda_1)\, \dots\, B(\lambda_l)\, \Omega \,+$$
$$+ \sum_{k=1}^{l} M_k(\lambda, \{\lambda\})\, B(\lambda_1)\, \dots\, \widehat{B}(\lambda_k)\, \dots\, B(\lambda_l)\, B(\lambda)\, \Omega \quad . \tag{88}$$

The first term in RHS has a desirable form and is obtained using only the first term in the RHS of (67). All other terms are combinations of $2^l - 1$ terms, which one has taking $A(\lambda)$ to Ω using the exchange relation (67). The coefficients M_k can be quite involved. However the coefficient M_1 is simple enough, to get it one must use the second term in (67) during the interchange of $A(\lambda)$ and $B(\lambda_1)$ and in all other exchanges use only the first term in RHS of (67). Thus we get

$$M_1(\lambda, \{\lambda\}) \,=\, g(\lambda - \lambda_1) \prod_{k=2}^{l} f(\lambda_1 - \lambda_k)\, \alpha^N(\lambda_1)\,. \tag{89}$$

Now we comment, that due to the commutativity of $B(\lambda)$ all other coefficients $M_j(\lambda, \{\lambda\})$ are obtained from $M_1(\lambda, \{\lambda\})$ by a simple substitution $\lambda_1 \rightarrow \lambda_j$ so that

$$M_j(\lambda, \{\lambda\}) \,=\, g(\lambda - \lambda_j) \prod_{k\neq j}^{l} f(\lambda_j - \lambda_k)\, \alpha^N(\lambda_j)\,. \tag{90}$$

This means, of course, that the coefficients $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, entering R–matrix, satisfy involved sum rules, making the FCR consistent.

Analogously for $D(\lambda)$ we have

$$D(\lambda)\, B(\lambda_1) \,\ldots\, B(\lambda_l)\, \Omega \,=\, \prod_{k=1}^{l} h(\lambda - \lambda_k)\, \delta^N(\lambda)\, B(\lambda_1) \,\ldots\, B(\lambda_l)\, \Omega \,+$$
$$+ \sum_{k=1}^{l} N_k(\lambda, \{\lambda\})\, B(\lambda_1) \,\ldots\, \widehat{B}(\lambda_k) \,\ldots\, B(\lambda_l)\, B(\lambda)\, \Omega \,\,, \tag{91}$$

where

$$N_j(\lambda, \{\lambda\}) \,=\, k(\lambda - \lambda_j) \prod_{k\neq j}^{l} h(\lambda_j - \lambda_k)\, \delta^N(\lambda_j)\,. \tag{92}$$

Observe now, that

$$g(\lambda - \lambda_j) \,=\, -k(\lambda - \lambda_j) \,\,. \tag{93}$$

This allows to cancel the unwanted terms in (88) and (91) for the application of $A(\lambda) \,+\, D(\lambda)$ to $\Phi(\{\lambda\})$. We get, that

$$(A(\lambda) \,+\, D(\lambda)\,)\, \Phi(\{\lambda\}) \,=\, \Lambda(\lambda, \{\lambda\})\, \Phi(\{\lambda\}) \tag{94}$$

with

$$\Lambda(\lambda, \{\lambda\}) \,=\, \alpha^N(\lambda) \prod_{j=1}^{l} f(\lambda - \lambda_j) \,+\, \delta^N(\lambda) \prod_{j=1}^{l} h(\lambda - \lambda_j)\,, \tag{95}$$

if the set of $\{\lambda\}$ satisfy the equations

$$\prod_{k\neq j}^{l} f(\lambda_j - \lambda_k)\, \alpha^N(\lambda_j) \,=\, \prod_{k\neq j}^{l} h(\lambda_j - \lambda_k)\, \delta^N(\lambda_j) \tag{96}$$

for $j \,=\, 1, \ldots, l$. Using the explicit expressions (69) and (85) we rewrite (96) in the form

$$\left(\frac{\lambda_j + i/2}{\lambda_j - i/2}\right)^N \,=\, \prod_{k\neq j}^{l} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} \quad. \tag{97}$$

This is the main result of this section. In what follows we shall use the equations (97) to investigate the $N \to \infty$ limit.

An important observation is, that equations (97) mean, that the superficial poles in the eigenvalue $\Lambda(\lambda, \{\lambda\})$ actually cancel so that Λ is a polynomial in λ of degree N as it should. This observation makes one think that only solutions $\{\lambda\}$ with $\lambda_j \neq \lambda_k$ are relevant for our purpose. Indeed, equal λ will lead to higher order spurious poles, cancelling of which requires more than l equations. And indeed we shall see below, that solutions with nonequal λ_j are enough to give all spectrum.

The equations (97) appeared first (in a different form) in the paper of H.Bethe in 1931, in which exactly the hamiltonian H was investigated. The algebraic derivation in this lecture is completely different from the original approach of Bethe, who used an explicit Ansatz for the eigenvectors Φ in a concrete coordinate representation for the spin operators. The term Bethe Ansatz originates from that paper. We propose to call our approach "The Algebraic Bethe Ansatz" (ABA). The equations (97) and vector $\Phi(\{\lambda\})$ will be called Bethe Ansatz equations (BAE) and Bethe vector correspondingly.

We finish this section by giving the explicit expressions for the eigenvalues of the important observables on Bethe vectors. We begin with the spin.

Taking limit $\mu \to \infty$ in FCR (44) and using (45) we get the following relation

$$\left[T_a(\lambda) \, , \, \frac{1}{2}\sigma^\alpha + S^\alpha \right] = 0 \quad , \tag{98}$$

which expresses the $sl(2)$ invariance of the monodromy in the combined space $\mathcal{H} \otimes V$. From here we have in particular

$$[\, S^3 \, , \, B \,] = -B \qquad , \tag{99}$$

$$[\, S^+ \, , \, B \,] = A - D \quad . \tag{100}$$

Now for reference state Ω we have

$$S^+ \Omega = 0 \, , \quad S^3 \Omega = \frac{N}{2} \Omega \ , \tag{101}$$

showing, that it is the highest weight for spin S^α.

From (99) and (101) we have

$$S^3 \Phi(\{\lambda\}) = \left(\frac{N}{2} - l \right) \Phi(\{\lambda\}) \ . \tag{102}$$

Let us show, that

$$S^+ \Phi(\{\lambda\}) = 0 \ . \tag{103}$$

From (100) we have

$$\begin{aligned} S^+ \Phi(\{\lambda\}) &= \sum_j B(\lambda_1) \dots B(\lambda_{j-1})(A(\lambda_j) - D(\lambda_j)) B(\lambda_{j+1}) \dots B(\lambda_l) \Omega \\ &= \sum_k O_k(\{\lambda\}) B(\lambda_1) \dots \widehat{B}(\lambda_k) \dots B(\lambda_l) \Omega \end{aligned} \tag{104}$$

and repeating the procedure that was used to derive BAE we can show, that all coefficients $O_k(\{\lambda\})$ vanish if BAE are satisfied.

Thus $\Phi(\{\lambda\})$ are all highest weights. In particular it means, that the l cannot be too large, because the S^3 eigenvalue of the highest weight is nonnegative. More exactly we have an estimate

$$l \leq \frac{N}{2} \quad . \tag{105}$$

We see that the cases of even and odd N are quite different. When N is even, the spin of all states is integer and there are $sl(2)$ invariant states, corresponding to $l = N/2$. For odd N spins are half–integer.

Now we turn to the shift operator. For $\lambda = i/2$ the second term (and many of its derivatives over λ) in $\Lambda(\lambda, \{\lambda\})$ vanishes and this eigenvalue becomes multiplicative. In particular

$$U\,\Phi(\{\lambda\}) = i^N F(i/2)\Phi(\{\lambda\}) = \prod_j \frac{\lambda_j + i/2}{\lambda_j - i/2}\Phi(\{\lambda\}) \,. \tag{106}$$

Taking log here we see, that the eigenvalues of the momentum P are additive and

$$P\,\Phi(\{\lambda\}) = \sum_j p(\lambda_j)\,\Phi(\{\lambda\}) \quad , \tag{107}$$

where

$$p(\lambda) = \frac{1}{i}\ln\frac{\lambda + i/2}{\lambda - i/2} \quad . \tag{108}$$

The additivity property holds also for the energy H. Differentiating $\ln\Lambda$ over λ once and putting $\lambda = i/2$ we get

$$H\,\Phi(\{\lambda\}) = \sum_j \epsilon(\lambda_j)\,\Phi(\{\lambda\}) \quad , \tag{109}$$

where

$$\epsilon(\lambda) = -\frac{1}{2}\frac{1}{\lambda^2 + 1/4} \quad . \tag{110}$$

Formulas (108) and (110) allow to use the quasiparticle interpretation for the spectrum of observables on Bethe vectors. Each quasiparticle is created by operator $B(\lambda)$, it diminishes the S^3 eigenvalue by 1 and has momentum $p(\lambda)$ and energy $\epsilon(\lambda)$ given in (108) and (110). Let us note, that

$$\epsilon(\lambda) = \frac{1}{2}\frac{d}{d\lambda}p(\lambda) \quad . \tag{111}$$

The variable λ in this interpretation can be called a rapidity of a quasiparticle.

It is possible to exclude the rapidity to get the dispersion relation, describing connection of energy and momentum

$$\epsilon(p) = \cos p - 1 \quad . \tag{112}$$

The eigenvalues of hamiltonian are all negative, so that the reference state Ω cannot be taken as a ground state, i.e. state of the lowest energy. It trivially changes if we take $-H$ as a hamiltonian. Both cases H and $-H$ are interesting for the physical applications, corresponding to antiferromagnetic and ferromagnetic phases, correspondingly. The mathematical (and physical) features of the $N \to \infty$ limit in these two cases are completely different, as we shall see soon.

5 $\mathrm{XXX}_{1/2}$ model. Physical spectrum in the ferromagnetic thermodynamic limit

In our case the thermodynamic limit is just limit $N \to \infty$. We shall see, how BAE simplify in this limit. Looking at BAE (97) we see that N enters there only in the exponent in the LHS. For real $\lambda_1, \ldots, \lambda_l$ both sides in BAE are functions with values on the circle and LHS is wildly oscillating when N is large. Taking the log we get

$$Np(\lambda_j) = 2\pi Q_j + \sum_{k=1}^{l} \varphi(\lambda_j - \lambda_k) \,, \tag{113}$$

where the integers Q_j, $0 \leq Q_j \leq N-1$ define the branch of the log and $\varphi(\lambda)$ is a fixed branch of $\ln \frac{\lambda+i}{\lambda-i}$. For large N and Q and fixed l the second term in the RHS of (113) is negligible and we get the usual quasicontinuous expression for the momentum of a free particle on a chain

$$p_j = 2\pi \frac{Q_j}{N} \,. \tag{114}$$

In the ferromagnetic case when the hamiltonian is $-H$ the energy of this particle is given by $\epsilon(p) = 1 - \cos p$.

The correction (second term in RHS of (113)) expresses the scattering of these particles. The comparison with the usual quantum mechanical treatment of a particle in a box shows that $\varphi\,(\lambda_i - \lambda_k)$ plays the role of the phase shift of particles with rapidities λ_j and λ_k. Thus the function

$$S(\lambda - \mu) = \frac{\lambda - \mu + i}{\lambda - \mu - i} \tag{115}$$

is a corresponding S–matrix element.

Another analogy is with the classical inverse scattering method, where the combination

$$Z(\lambda) \;=\; B(\lambda)\, A^{-1}(\lambda) \tag{116}$$

was more important, than $B(\lambda)$. The factor S enters the exchange algebra

$$Z(\lambda)\, Z(\mu) \;=\; Z(\mu)\, Z(\lambda)\, S(\lambda - \mu) \,, \tag{117}$$

valid in the limit $N \to \infty$ because the second term in the RHS of (67) effectively vanishes. Operator $Z(\lambda)$ can be interpreted as a creation operator of a normalized particle state.

This argument by analogy should be justified by the adequate scattering theory applicable to our case. We do not have time to do it here and refer to the original papers of Babbit and Thomas.

The BAE allow also for the complex solutions which in our situation correspond to bound states. Let us see it in more detail. The first nontrivial case is $l = 2$. From two BAE

$$\left(\frac{\lambda_1 + i/2}{\lambda_1 - i/2}\right)^N = \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i} \,, \tag{118}$$

$$\left(\frac{\lambda_2+i/2}{\lambda_2-i/2}\right)^N=\frac{\lambda_2-\lambda_1+i}{\lambda_2-\lambda_1-i} \tag{119}$$

we see, that

$$\left(\frac{\lambda_1+i/2}{\lambda_1-i/2}\right)^N\left(\frac{\lambda_2+i/2}{\lambda_2-i/2}\right)^N=1\;, \tag{120}$$

so that $p(\lambda_1)+p(\lambda_2)$ is real. Further, for $\mathrm{Im}\lambda_1\neq 0$ the LHS in (118) grows (or decreases) exponentially when $N\to\infty$ and to compensate it in the RHS we must have, that

$$\mathrm{Im}(\lambda_1-\lambda_2)=i\quad(\text{or }\;-i)\;. \tag{121}$$

As λ_1 and λ_2 can be interchanged, we can say, that in the limit $N\to\infty$ λ_1 and λ_2 acquire the form

$$\lambda_1=\lambda_{1/2}+\frac{i}{2}\,,\quad \lambda_2=\lambda_{1/2}-\frac{i}{2}\;\;, \tag{122}$$

where $\lambda_{1/2}$ is real. In the thermodynamic limit $\lambda_{1/2}$ can become arbitrary.

The momentum $p_{1/2}(\lambda)$ and energy $\epsilon_{1/2}(\lambda)$ for the corresponding Bethe vector are given by

$$e^{ip_{1/2}(\lambda)}=e^{ip_0(\lambda+i/2)+ip_0(\lambda-i/2)}=\frac{\lambda+i/2+i/2}{\lambda-i/2+i/2}\cdot\frac{\lambda+i/2-i/2}{\lambda-i/2-i/2}=\frac{\lambda+i}{\lambda-i} \tag{123}$$

and

$$\epsilon_{1/2}(\lambda)=\frac{1}{2}\frac{d}{d\lambda}\ln p_{1/2}(\lambda)=\frac{1}{\lambda^2+1}\;. \tag{124}$$

The origin of notation $p_0(\lambda)$ for the momentum (108) and $p_{1/2}(\lambda)$, $\epsilon_{1/2}(\lambda)$ for momentum and energy of the complex solution will be clear soon. Excluding λ from (123) and (124) we get

$$\epsilon_{1/2}(p)=\frac{1}{2}(1-\cos p)\;. \tag{125}$$

The interpretation of this eigenvector as bound state is supported by the inequality

$$\epsilon_{1/2}(p)<\epsilon_0(p-p_1)+\epsilon_0(p_1) \tag{126}$$

for all p, p_1, $\;0\leq p,\,p_1\leq 2\pi$. The S–matrix elements for scattering will be discussed later.

For $l>2$ the complex solutions are described analogously. Roots λ_l are combined in the complexes of type M, where M runs through half–integer values $M=0,1/2,1,\ldots$, defining the partition

$$l=\sum_M \nu_M(2M+1)\;, \tag{127}$$

where ν_M gives the number of complexes of type M.

The set of integers $\{\nu_M\}$ defines a configuration of Bethe roots. Each complex contains roots of the type

$$\lambda_{M,m}=\lambda_M+im\;,\quad -M\leq m\leq M\;, \tag{128}$$

where λ_M is real, m being integer or half–integer together with M. The momentum and energy of complex is given by

$$p_M(\lambda) = \frac{1}{i} \ln \frac{\lambda + i(M+1/2)}{\lambda - i(M+1/2)} \tag{129}$$

and

$$\epsilon_M(\lambda) = \frac{1}{2} \frac{2M+1}{\lambda^2 + (M+1/2)^2} = \frac{1}{2M+1}(1 - \cos p_M) \ . \tag{130}$$

The S–matrix element for the scattering of complex of type 0 on a complex of type M is given by

$$S_{0,M}(\lambda) = \frac{\lambda + iM}{\lambda - iM} \cdot \frac{\lambda + i(M+1)}{\lambda - i(M+1)} \tag{131}$$

and for scattering of complexes M and N

$$S_{M,N}(\lambda) = \prod_{L=|M-N|}^{M+N} S_{0,L}(\lambda) \ . \tag{132}$$

It is the superficial analogy of this formula with the Klebsch–Gordan formula for $sl(2)$ which prompted me to use label M for complexes instead of their length $2M+1$.

The derivations are just the direct calculation of products

$$\prod_{m=-M}^{M} \frac{\lambda + i/2 + im}{\lambda - i/2 + im}, \quad \prod_{m=-M}^{M} \frac{\lambda + i + im}{\lambda - i + im}, \quad \prod_{m,n=-M,N}^{M,N} \frac{\lambda + i + i(m+n)}{\lambda - i + i(m+n)},$$

where many terms cancel.

This finishes the description of the physical spectrum of $-H$ in the thermodynamic limit. The ground state $\Omega = \prod \omega_n$ defines the incomplete infinite tensor product in the sense of John von Neumann. The space $\mathcal{H}_{\mathrm{F}}$ is a completion of the states which differ from ω_n only in finite number of factors in $\prod \otimes h_n$. The excitations are particles, classified by half–integers M, $M = 0, 1/2, 1, \ldots$ and rapidity λ (or momentum p). The dispersion law for a particle of type M is given by (130) and scattering matrix elements are given by (132). The interpretation of particles with $M > 0$ as bound states is possible but not necessary.

The spin components $S^{\pm}$ have no sense in the physical Hilbert space $\mathcal{H}_{\mathrm{F}}$ as they change vector Ω in any site n. Operator S^3 after shift by $N/2$

$$Q = \frac{N}{2} - S^3 \tag{133}$$

makes sense in $\mathcal{H}_{\mathrm{F}}$ and has integer eigenvalues $2M+1$. This phenomenon gives the example of "symmetry breaking" by vacuum: only $U(1)$ part of $sl(2)$ remains intact in the thermodynamic limit. Physically we call this phase ferromagnetic because the prescribed direction of spin (in 3-d direction in spin space) is macroscopic and fixed.

6 $\mathrm{XXX}_{1/2}$ model. BAE for an arbitrary configuration

Before turning to physics of the antiferromagnetic chain we shall consider in more detail the BAE for arbitrary configuration $\{\nu_M\}$, where each integer ν_M gives the number of complexes of type M combined from l quasiparticles, so that

$$l = \sum_M (2M+1)\nu_M \quad . \tag{134}$$

We shall investigate the approximate BAE for the real centers of complexes $\lambda_{M,i}$, and allow l to be of order $N/2$. There are some doubts, expressed in the literature, about validity of the picture of complexes in this case. Indeed, in our arguing above we supposed, that l is much smaller than N. However in the physical applications only ν_0 will be large and in this case the picture of complexes is correct.

The BAE for the $\lambda_{M,j}$, $j = 1, \ldots, \nu_M$ are obtained by multiplying the BAE for each complex in the LHS of (97) and rearranging the RHS according to the picture of complexes. They look as follows

$$e^{ip_M(\lambda_{M,j})N} = \prod_{M'} \prod_{(M',k)\neq(M,j)} S_{M,M'}(\lambda_{M,j} - \lambda_{M',k}). \tag{135}$$

The factors in the RHS are scattering matrix elements from (132); in the LHS the momentum $p_M(\lambda)$ from (129) enters; condition $(M', k) \neq (M, j)$ means, that among $\nu_{M'}$ roots $\lambda_{M',k}$, entering the RHS in (135), the one which is equal to $\lambda_{M,j}$ from LHS is absent.

Taking the logarithm of (135) and using the basic branch in the form

$$\frac{1}{i} \ln \frac{\lambda + ia}{\lambda - ia} = \pi - 2\mathrm{arctg}\frac{\lambda}{a} \tag{136}$$

we get the equation

$$2N\mathrm{arctg}\frac{\lambda_{M,j}}{M+1/2} = 2\pi Q_{M,j} + \sum_{M'} \sum_{(M',k)\neq(M,j)} \Phi_{M,M'}(\lambda_{M,j} - \lambda_{M',k}) \ , \tag{137}$$

where

$$\Phi_{M,M'}(\lambda) = 2 \sum_{L=|M-M'|}^{M+M'} \left(\mathrm{arctg}\frac{\lambda}{L} + \mathrm{arctg}\frac{\lambda}{L+1} \right) \tag{138}$$

with the understanding that the term with $L = 0$ is omitted and $Q_{M,j}$ is an integer or half–integer (depending on the configuration, which parametrizes the roots $\lambda_{M,j}$).

The main hypothesis of our investigation is that $Q_{M,j}$ classify the roots uniquely and monotonously: roots $\lambda_{M,j}$ increase, when $Q_{M,j}$ increase; moreover there are no coinciding $Q_{M,j}$ for a given complex of type M. The last condition corresponds to the requirement for the BAE roots to be distinct, which was discussed in the course of derivation of BAE.

We shall look for the real and bounded solutions of equation (137). For those the numbers $Q_{M,j}$ have a natural bound. Indeed, taking into account, that

$$\mathrm{arctg} \pm \infty = \pm\frac{\pi}{2} \tag{139}$$

and putting $\lambda_{M,j} = \infty$ we get for the corresponding $Q_{M,j}$ the expression

$$Q_{M,\infty} = -\sum_{M' \neq M} (2\min(M, M') + 1)\,\nu_{M'} - \left(2M + \frac{1}{2}\right)(\nu_M - 1) + \frac{N}{2}\,. \tag{140}$$

The maximal admissible $Q_{M,j}$ is then

$$Q_{M,\max} = Q_{M,\infty} - (2M + 1) \tag{141}$$

because complex of type M has $(2M + 1)$ roots. We suppose, that when $Q_{M,j}$ gets values bigger than $Q_{M,\max}$, the roots in our complex turn to be infinite one after another, so that for $Q_{M,j} = Q_{M,\infty}$ the whole complex becomes infinite.

I understand, that all this is quite a host of hypotheses, but the result we shall get soon is quite satisfactory. It will be nice to produce more detailed justification for our considerations.

From (140) and (141) we get

$$Q_{M,\max} = \frac{N}{2} - \sum_{M'} J(M, M')\nu_{M'} - \frac{1}{2}\ \ , \tag{142}$$

where

$$J(M, M') = \begin{cases} 2\min(M, M') + 1 & M \neq M' \\ 2M + \frac{1}{2} & M = M'\ . \end{cases} \tag{143}$$

Analogously we find $Q_{M,\min}$. Due to the fact, that $\operatorname{arctg}\lambda$ is odd we have

$$Q_{M,\min} = -Q_{M,\max}\ \ . \tag{144}$$

Thus for the number of vacances P_M for the numbers $Q_{M,j}$ we have

$$P_M = 2Q_{M,\max} + 1 = N - 2\sum_{M'} J(M, M')\nu_{M'}\ \ . \tag{145}$$

The numbers $Q_{M,j}$ are integers for odd P_M and half–integers for P_M even.

Now we can estimate the number of Bethe vectors, characterized by the admissible numbers $Q_{M,j}$. For a given configuration $\{\nu_M\}$ the states are given by fixing the distribution of Q-s over the vacances; so the whole number $Z(N, \{\nu_M\})$ of them is given by

$$Z(N, \{\nu_M\}) = \prod_M C_{P_M}^{\nu_M}\ \ , \tag{146}$$

where C_n^m is a binomial coefficient

$$C_n^m = \frac{n!}{m!(n-m)!}\ \ . \tag{147}$$

Let us consider the number of states for given l and number of complexes

$$q = \sum \nu_M \tag{148}$$

inside each configuration $\{\nu_M\}$

$$Z(N, l, q) = \sum_{\substack{\sum(2M+1)\nu_M = l \\ \sum \nu_M = q}} Z(N; \{\nu_M\})\ \ . \tag{149}$$

We shall calculate $Z(N,l,q)$ by reduction via a partial summation. For that we shall begin by extracting the contribution of roots of type 0, or in other words by substituting the configuration $\{\nu_M\}$ by $\{\nu'_M\}$, where

$$\nu'_M = \nu_{M+1/2}, \quad M = 0, \frac{1}{2}, \dots \quad . \tag{150}$$

First we observe, that

$$P_M(N, \{\nu_M\}) = P_{M-1/2}\left(N - 2q, \{\nu'_M\}\right) \quad . \tag{151}$$

Indeed, it is easy to see that

$$J(M, M') = J(M - \frac{1}{2}, M' - \frac{1}{2}) + 1 \quad , \tag{152}$$

so that for $M \geq 1/2$

$$\begin{aligned} P_M(N, \{\nu_M\}) &= N - 2J(M,0)\nu_0 - 2\sum_{M' \geq 1/2} J(M, M')\nu_{M'} = \\ &= N - 2\nu_0 - 2\sum_{M' \geq 1/2} \left(J\left(M - \frac{1}{2},\ M' - \frac{1}{2}\right) + 1\right)\nu_{M'} = \\ &= N - 2q - 2\sum_{M'} J\left(M - \frac{1}{2}, M'\right)\nu'_{M'} \end{aligned} \tag{153}$$

and (151) follows. Thus we have a recurrence relation

$$Z(N, \{\nu_M\}) = C_{P_0}^{\nu_0} Z\left(N - 2q,\ \{\nu'_M\}\right) \tag{154}$$

and summing over the allowed ν_0 we get

$$Z(N,l,q) = \sum_{\nu=0}^{q-1} C_{N-2q+\nu}^{\nu} Z(N - 2q,\ l - q,\ q - \nu) \ . \tag{155}$$

With the initial condition

$$Z(N, 1, 1) = N - 1 \tag{156}$$

this gives

$$Z(N,l,q) = \frac{N - 2l + 1}{N - l + 1} C_{N-l+1}^{q} C_{l-1}^{q} \tag{157}$$

and finally for the number of the Bethe vectors with given l

$$Z(N,l) = \sum_{q=1}^{l} Z(N,l,q) = C_N^l - C_N^{l-1} \quad . \tag{158}$$

Now we remember that each Bethe vector of spin $\frac{N}{2} - l$ is a highest weight in the multiplet of dimension $N - 2l + 1$. Thus the full number of states, described in our picture

$$Z = \sum_l (N - 2l + 1) Z(N,l) = 2^N \tag{159}$$

is equal to the dimension of our Hilbert space. This is very satisfactory and strongly confirms all the hypotheses, which we used in this calculation. We stop here the general investigation of the BAE (137).

7 $\mathrm{XXX}_{1/2}$ model. Physical spectrum in the antiferromagnetic case

I am ready now to describe some important states. The ground state, i.e. the state of the lowest energy, is obtained by taking the maximal number of real roots. We shall suppose, that N is even to be able to have an $sl(2)$ invariant state. The corresponding configuration looks like

$$\nu_0 = \frac{N}{2}\,; \;\; \nu_M = 0\,, \;\; M \geq \frac{1}{2}. \tag{160}$$

For this configuration $l = N/2$ and so

$$S^3 = \frac{N}{2} - l = 0 \;\; ; \tag{161}$$

thus the spin of the state vanishes. Now the number of vacances P_0

$$P_0 = N - 2J(0,0)\nu_0 = N - \frac{N}{2} = \frac{N}{2} \tag{162}$$

is equal to the number of roots and so there is no freedom for the allocating the numbers $Q_{0,k}$; they span all interval

$$-\frac{N}{4} + \frac{1}{2} \leq Q_{0,k} \leq \frac{N}{4} - \frac{1}{2} \tag{163}$$

being integer (half–integer) for $N/2$ odd (even).

Thus the state in question is unique and gives us the singlet for $sl(2)$ group of spin observables.

Using physical terminology we can call this state the Dirac sea of quasiparticles. The mere existence of this state is due to the Fermi character of the quasiparticles spectrum, i.e. to the condition that all $Q_{0,j}$ are distinct.

In what follows we consider states, for which ν_0 differs from its maximal value $N/2$ by a finite amount

$$\nu_0 = \frac{N}{2} - \kappa \;\; . \tag{164}$$

We shall see, how a Fock-like space of excitations will emerge step by step with increasing of κ in the limit $N \to \infty$. Thus it is κ, which will play the role of the "grading" in our definition of the physical portion in the formal infinite tensor product $\prod \otimes \mathrm{C}^2$.

For a fixed κ all ν_M, $M \geq 1/2$ are bounded, when $N \to \infty$. Indeed, from inequality $l \leq N/2$ it follows

$$\sum_{M \geq 1/2} (2M+1)\nu_M = l - \nu_0 \leq \frac{N}{2} - \nu_0 = \kappa \tag{165}$$

and N disappears from this estimate. Thus for a fixed κ the number of configurations is finite and we can consider them one after another.

For $\kappa = 1$ only $\nu_M = 0$, $M \geq 1/2$ are allowed. The l for these states is $l = N/2-1$, thus the spin is 1. The number of vacances

$$P_0 = N - 2 \cdot \frac{1}{2}\left(\frac{N}{2} - 1\right) = \frac{N}{2} + 1 \tag{166}$$

exceeds the number of roots by two. Thus two admissible numbers $Q_{0,j}$ are not to be used, which gives the two parameter degeneracy of the state. In physical jargon one speaks of the holes in the Dirac sea.

For $\kappa = 2$ there are two possibilities: $\nu_M = 0$, $M \geq 1/2$ and $\nu_{1/2} = 1$, $\nu_M = 0$, $M \geq 1$. Let us consider the latter in more detail. First, $l = N/2$ for it, so that the spin vanishes. Second, for the corresponding P_0 we have

$$P_0 = N - 2\left(\frac{N}{2} - 2\right) \cdot \frac{1}{2} - 2J\left(0, \frac{1}{2}\right) = \frac{N}{2} \tag{167}$$

and

$$P_{1/2} = N - 2\left(\frac{N}{2} - 2\right) J\left(\frac{1}{2}, 0\right) - 2J\left(\frac{1}{2}, \frac{1}{2}\right) = 4 - 3 = 1\ . \tag{168}$$

We see, that the number of vacances for real roots once more exceeds their number by 2 and there is no freedom at all for the root of complex of type 1/2.

In the former case

$$P_0 = \frac{N}{2} - 2\left(\frac{N}{2} - 2\right)\frac{1}{2} = \frac{N}{2} + 2 \tag{169}$$

exceeds the number of roots by 4 and the spin of the state is 2.

For general κ we have a similar picture. First, for the configurations

$$\nu_0 = \frac{N}{2} - \kappa; \quad \nu_M = 0, \quad M \geq 1/2 \tag{170}$$

we have

$$P_0 = \frac{N}{2} + \kappa \quad , \tag{171}$$

so that there are 2κ holes, characterized by the missing places in the choice of admissible $Q_{0,j}$. The spin of this state is equal to κ. Then there are states with a smaller spin, corresponding to a few nonzero ν_M, $M \geq 1/2$. More on this will be said later.

We return to a more detailed characteristic of the states already described. For this more control over the roots of BAE is needed. Fortunately these equations are simplified drastically in the thermodynamic limit $N \to \infty$. The real roots become quasicontinuous and we can evaluate their distribution.

We begin with the ground state. The roots are real and corresponding $Q_{0,j}$ fill without holes all interval (163). We shall put for $N/2$ odd (with an evident correction for $N/2$ even)

$$Q_{0,j} = j \ , \tag{172}$$

so that the BAE take the form

$$\mathrm{arctg} 2\lambda_j = \frac{\pi j}{N} + \frac{1}{N}\sum_k \mathrm{arctg}(\lambda_j - \lambda_k). \tag{173}$$

The variable

$$x = \frac{j}{N} \tag{174}$$

becomes continuous in the limit $N \to \infty$ with values $-1/4 \le x \le 1/4$; the set of roots λ_j turn into function $\lambda(x)$.

The equation (173) becomes

$$\text{arctg}2\lambda(x) = \pi x + \int_{-1/4}^{1/4} \text{arctg}\left(\lambda(x) - \lambda(y)\right) dy \tag{175}$$

and looks rather formidable. Fortunately it is not $\lambda(x)$ which is of prime concern to us. Indeed, we are interested in the eigenvalues of local observables, which take the form of the sums over roots (see e.g. (107)). With our conventions we have

$$\sum_j h(\lambda_j) = N \int_{-1/4}^{1/4} h(\lambda(x))dx = N \int_{-\infty}^{\infty} h(\lambda)\rho(\lambda)d\lambda \ , \tag{176}$$

where the change of variables $\lambda: \ x \to \lambda(x)$ maps interval $-1/4 \le x \le 1/4$ into whole real line $-\infty < \lambda < \infty$ due to the monotonicity of $\lambda(x)$. The density $\rho(\lambda)$ is nothing but

$$\rho(\lambda) = \frac{dx}{d\lambda} = \frac{1}{\lambda'(x)}\bigg|_{x=\lambda^{-1}(\lambda)} \ . \tag{177}$$

Differentiating (175) we get for this density, which we denote by $\rho_0(\lambda)$ for our state, a linear integral equation

$$\frac{2}{1+4\lambda^2} = \pi\rho_0(\lambda) + \int_{-\infty}^{\infty} \frac{\rho_0(\mu)}{1+(\lambda-\mu)^2} d\mu \ , \tag{178}$$

which can be easily solved by Fourier transform. We get

$$\rho_0(\lambda) = \frac{1}{2\cosh \pi\lambda} \ . \tag{179}$$

The momentum and energy of the ground state are given by

$$P_0 = N \int p_0(\lambda)\rho_0(\lambda)d\lambda = 0 \tag{180}$$

due to the fact, that the integrand is odd if we use for $p_0(\lambda)$ slightly shifted expression

$$p_0(\lambda) = -2\text{arctg}2\lambda \tag{181}$$

and

$$E_0 = N \int \epsilon_0(\lambda)\rho_0(\lambda)d\lambda = -N\ln 2 \ . \tag{182}$$

Note, that the sign of $\epsilon_M(\lambda)$ in this section is opposite to that in section 5. Thus the energy of the ground state is proportional to the volume as always in the correct thermodynamic limit. Adding it to the hamiltonian H will make it nonnegative.

Now we turn to the configuration $\nu_0 = N/2 - 1, \ \nu_M = 0, \ M \ge 1/2$. There are two holes and we can put

$$Q_{0,j} = j + \theta(j - j_1) + \theta(j - j_2) \ , \tag{183}$$

where θ is a step function

$$\theta(j) = \begin{cases} 1 & j \geq 0 \\ 0 & j < 0 \end{cases} \tag{184}$$

and j_1 and j_2 are integer, characterizing the holes. By the same trick as before we get for the distribution $\rho_{\rm t}(\lambda)$ (t for triplet) of real roots the linear integral equations

$$\frac{2}{1+4\lambda^2} = \pi\rho_{\rm t}(\lambda) + \int_{-\infty}^{\infty} \frac{\rho_{\rm t}(\mu)}{1+(\lambda-\mu)^2} d\mu + \frac{\pi}{N}(\delta(\lambda-\lambda_1)+\delta(\lambda-\lambda_2)) \ , \tag{185}$$

where λ_1 and λ_2 are images of $x_1 = j_1/N$ and $x_2 = j_2/N$ in the map $x \to \lambda(x)$ defined by $\lambda_{\rm t}(x)$ (or $\lambda_0(x)$, because λ_1 and λ_2 enter in terms of order $1/N$). From (185) we get

$$\rho_{\rm t}(\lambda) = \rho_0(\lambda) + \frac{1}{N}(\sigma(\lambda-\lambda_1)+\sigma(\lambda-\lambda_2)) \ \ , \tag{186}$$

where $\sigma(\lambda)$ solves the equation

$$\sigma(\lambda) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma(\mu)}{1+(\lambda-\mu)^2} d\mu + \delta(\lambda) = 0 \ \ . \tag{187}$$

Solving it we can evaluate the momentum and energy of the corresponding state

$$P = k(\lambda_1) + k(\lambda_2) \ \ ; \tag{188}$$

$$E = E_0 + h(\lambda_1) + h(\lambda_2) \ \ , \tag{189}$$

where

$$k(\lambda) = \mathrm{arctg}\sinh\pi\lambda, \quad \epsilon(\lambda) = \frac{\pi}{2\cosh\pi\lambda} \ \ . \tag{190}$$

It is time to comment, that the constructed states once more allow for the particle interpretation: we have described a family of two particle states with the energy and momentum of a particle given by (190) and dispersion law is

$$\epsilon(k) = \frac{\pi}{2}\cos k, \qquad -\pi/2 \leq k \leq \pi/2 \ \ . \tag{191}$$

Next example is $\nu_0 = N/2-2$, $\nu_{1/2} = 1$, $\nu_M = 0$, $M \geq 1$. For the density of real roots $\rho_{\rm s}(\lambda)$ (s for singlet) we get the equation

$$\begin{aligned} \frac{2}{1+4\lambda^2} &= \pi\rho_{\rm s}(\lambda) + \int_{-\infty}^{\infty} \frac{\rho_{\rm s}(\mu)}{1+(\lambda-\mu)^2} d\mu + \\ &+ \frac{1}{N}\Big(\delta(\lambda-\lambda_1)+\delta(\lambda-\lambda_2)+\Phi'_{0,1/2}(\lambda-\lambda_{1/2})\Big) \ , \end{aligned} \tag{192}$$

where λ_1 and λ_2 stand for the holes and the last term in the RHS is a contribution of the complex of type $1/2$. For $\lambda_{1/2}$ we have one more equation

$$\mathrm{arctg}\lambda_{1/2} = \frac{1}{N}\sum_j \Phi_{1/2,0}(\lambda_{1/2}-\lambda_{0,j}) \ \ , \tag{193}$$

because the corresponding number $Q_{1/2,j}$ has just one admissible value equal to zero.

From (192) we get

$$\rho_s(\lambda) = \rho_0(\lambda) + \frac{1}{N}(\sigma(\lambda-\lambda_1) + \sigma(\lambda-\lambda_2) + \omega(\lambda-\lambda_{1/2})) \ , \qquad (194)$$

where $\sigma(\lambda)$ is as above and ω is a solution of equation

$$\pi\omega(\lambda) + \int_{-\infty}^{\infty} \frac{\omega(\mu)}{1+(\lambda-\mu)^2} d\mu + \Phi'_{0,1/2}(\lambda) = 0 \ . \qquad (195)$$

To evaluate $\lambda_{1/2}$ let us rewrite (193) in the form

$$\begin{aligned} &\mathrm{arctg}\lambda_{1/2} - \int_{-\infty}^{\infty} \Phi_{1/2,0}(\lambda_{1/2}-\lambda)\rho_0(\lambda)d\lambda = \\ &= \frac{1}{N}\int_{-\infty}^{\infty} \Phi_{1/2,0}(\lambda_{1/2}-\lambda)[\sigma(\lambda-\lambda_1) + \sigma(\lambda-\lambda_2) + \omega(\lambda-\lambda_{1/2})]d\lambda \ , \end{aligned} \qquad (196)$$

where the limit $N \to \infty$ is already taken into account by changing the sums over $\lambda_{0,j}$ by integral over density $\rho_s(\lambda)$. LHS here vanishes for any $\lambda_{1/2}$ and contribution of ω disappears due to oddness of integrand ; so we get the equation

$$\int_{-\infty}^{\infty} \Phi_{1/2,0}(\lambda_{1/2}-\lambda)(\sigma(\lambda-\lambda_1) + \sigma(\lambda-\lambda_2))d\lambda = 0 \ , \qquad (197)$$

or

$$\mathrm{arctg}2(\lambda_{1/2}-\lambda_1) + \mathrm{arctg}2(\lambda_{1/2}-\lambda_2) = 0 \qquad (198)$$

with the solution

$$\lambda_{1/2} = \frac{\lambda_1+\lambda_2}{2} \ . \qquad (199)$$

We are ready now to evaluate the observables. The spin of our state is zero. For momentum and energy we get exactly the same expressions (188) and (189) as in the previous example; the contribution of a complex of type 1/2 cancels exactly.

The examples considered are all, which give a two-parameter family of states. Returning to the particle interpretation, we can say, that our particles have spin 1/2. Indeed, we constructed the highest weights in triplet and singlet two-particle states with exactly the same momentum and energy content. Thus they are highest weights in $\mathrm{C}^2 \otimes \mathrm{C}^2$ representation of a spin observable. This is why I say, that the particles have spin 1/2.

This picture is recurrently confirmed in the description of the next excitation. The state $\nu_0 = N/2 - \kappa$, $\nu_M = 0$, $M \geq 1/2$ defines a 2κ particle state being the highest weight in the highest spin irreducible component in $\prod^{2\kappa} \otimes \mathrm{C}^2$. All other states for the same κ are states of lower spin, entering into multiplets with the number of particles not exceeding 2κ. The contribution of complexes of type M into energy and momentum always vanishes, so that the energy-momentum expressions depend only on the number of particles.

All this allows to say, that the only excitation of our system is a particle with spin 1/2 and energy – momentum relation (191). Note, that the momentum $\kappa(\lambda)$ runs through the half of the usual Brillouin zone. There is one important restriction: the number of particles is even. These particles are usually referred to as spin waves. For a long time it was stated in the physical literature, that spin waves of the antiferromagnetic chain of spin 1/2 magnets has spin 1. Indeed, spin wave being a hole in the singlet Dirac sea corresponds to a turn of one spin, amounting to spin $1/2 + 1/2 = 1$.

However, our more precise analysis shows, that turn of a spin corresponds to 2 holes and two spin wave excitations. The momentum of this state runs through the whole Brillouin zone $-\pi \le k \le \pi$.

Mathematically the Hilbert space $\mathcal{H}_{\mathrm{AF}}$ is a (half) Fock space

$$\mathcal{H}_{\mathrm{AF}}^{\mathrm{even}} = \sum_{n=0}^{\infty} \int_{-\pi/2}^{\pi/2} d\kappa_1 \dots \int_{-\pi/2}^{\pi/2} d\kappa_{2n} \prod^{2n} \otimes \mathrm{C}^2 \ . \tag{200}$$

(Compare it with Professor's Miwa lectures, where $\mathcal{H}_{\mathrm{AF}}$ corresponds to $\Lambda_0 \otimes \Lambda_0$, or $\Lambda_1 \otimes \Lambda_1$).

Natural question is how to describe one particle (or odd number of particles) state. The answer is, that they enter the chain of odd length. The lowest energy state there has spin 1/2 and have one hole in the distribution of numbers $Q_{o,j}$. Thus this state becomes a one-particle state in the thermodynamic limit. Chain of odd length has no ground state but just 1 particle, 3 particles etc. states. The corresponding Hilbert space is

$$\mathcal{H}_{\mathrm{AF}}^{\mathrm{odd}} = \sum_{n=0}^{\infty} \int_{-\pi/2}^{\pi/2} d\kappa_1 \dots \int_{-\pi/2}^{\pi/2} d\kappa_{2n+1} \prod^{2n+1} \otimes \mathrm{C}^2 \tag{201}$$

(and corresponds to $\Lambda_0 \otimes \Lambda_1$ or $\Lambda_1 \otimes \Lambda_0$ in Professor's Miwa lectures).

We finish with some formalization of our result. The expression for the Bethe state

$$\Phi(\{\lambda\}) = B(\lambda_1) \dots B(\lambda_l)\Omega \tag{202}$$

can be formally rewritten as

$$\Phi(\{\lambda\}) = \left\{ \exp \sum_{\lambda_j=1}^{l} \ln B(\lambda_i) \right\} \Omega \tag{203}$$

and now it is possible to go to the thermodynamic limit. For the ground state Φ_0 we have

$$\Phi_0 = \exp \left\{ N \int_{-\infty}^{\infty} \ln B(\lambda) \rho_0(\lambda) d\lambda \right\} \Omega \ . \tag{204}$$

Exponential dependence of a true ground state on the volume is a typical phenomenon in quantum field theory. Now the triplet excited state $\Phi(\lambda_1, \lambda_2)$ can be written as

$$\Phi(\lambda_1, \lambda_2) = \tilde{Z}(\lambda_1)\tilde{Z}(\lambda_2)\Phi_0 \tag{205}$$

without any reference to Ω and dependence on the volume. Here

$$\tilde{Z}(\lambda) = \exp \left\{ \int \ln B(\lambda) \sigma(\lambda - \mu) d\mu \right\} \tag{206}$$

plays the role of a creation operator of one-particle state from the physical ground state.

As was mentioned in the section 5 it is

$$Z(\lambda) = \tilde{Z}(\lambda) A^{-1}(\lambda) \tag{207}$$

which is a more natural object in the scattering problem. Of course $Z(\lambda)$ is defined up to a constant normalization factor, which we can use to cancel the N-dependent factor $a_\infty(\lambda)$

$$a_\infty(\lambda) = (\lambda + \frac{i}{2})^N \exp\left\{ N \int_{-\infty}^{\infty} \ln \frac{\lambda - \mu - i}{\lambda - \mu} \rho_0(\mu) d\mu \right\} \tag{208}$$

entering the eigenvalue

$$A_N(\lambda)\Phi_0 = a_\infty(\lambda)\Phi_0 \tag{209}$$

in the limit $N \to \infty$, when the contribution of $D_N(\lambda)$ to $\Lambda(\lambda, \{\lambda\})$ vanishes.

Operators $Z(\lambda)$ satisfy the exchange relation

$$Z(\lambda)Z(\mu) = Z(\mu)Z(\lambda)S_{\mathrm{t}}(\lambda - \mu), \tag{210}$$

where the phase-factor $S_{\mathrm{t}}(\lambda - \mu)$ is given by

$$S_{\mathrm{t}}(\lambda) = \exp\left\{ \int_{-\infty}^{\infty} \ln \frac{\mu + i}{\mu - i} \sigma(\mu - \lambda) d\mu \right\} = \frac{1}{i} \frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(1 - \frac{i\lambda}{2})}{\Gamma(\frac{1-i\lambda}{2})\Gamma(1 + \frac{i\lambda}{2})} . \tag{211}$$

In course of derivation the contribution of the second term in the exchange relations (67) is negleqted, which can be justified in the limit $N \to \infty$.

The factor $S_{\mathrm{t}}(\lambda)$ is to be interpreted as a triplet eigenvalue of the S-matrix for spin 1/2 particles, acting in $\mathrm{C}^2 \otimes \mathrm{C}^2$

$$S^{1/2,1/2}(\lambda - \mu) = S_{\mathrm{t}}(\lambda - \mu) \left(\frac{\lambda - \mu}{\lambda - \mu + i} \mathrm{I} + \frac{i}{\lambda - \mu + i} \mathrm{P} \right) . \tag{212}$$

The creation operators $Z_\varepsilon(\lambda)$, $\varepsilon = \pm 1$ are to satisfy the exchange Zamolodchikov relation

$$Z_{\varepsilon_1}(\lambda_1) Z_{\varepsilon_2}(\lambda_2) = Z_{\varepsilon_2'}(\lambda_2) Z_{\varepsilon_1'}(\lambda_1) S_{\varepsilon_1 \varepsilon_2}^{\varepsilon_1' \varepsilon_2'}(\lambda - \mu) . \tag{213}$$

We did not construct operator $Z_\varepsilon(\lambda)$ (the vertex operators of the second kind in Professor Miwa terminology). We can only identify in $\mathcal{H}_{\mathrm{AF}}^{\mathrm{even}}$

$$Z_+(\lambda) Z_+(\mu) = Z(\lambda) Z(\mu) \quad , \tag{214}$$

where $Z(\lambda)$ is given by (207) and

$$\begin{aligned} & Z_+(\lambda)Z_-(\mu) - Z_-(\lambda)Z_+(\mu) = \\ & Z(\lambda) \exp\left\{ \int_{-\infty}^{\infty} \ln B(\sigma) \omega \left(\frac{\lambda + \mu}{2} - \sigma \right) d\sigma \right\} B\left(\frac{\lambda + \mu + i}{2} \right) B\left(\frac{\lambda + \mu - i}{2} \right) Z(\mu). \end{aligned} \tag{215}$$

This identification is, however, sufficient to justify all S-matrix (212).

The interesting but not understood comment on the formula (211) is as follows: the phase-factor $S_{\mathrm{t}}(\lambda)$ coincides with the S-matrix for the rotationally symmetric subspace of the Laplacian on Poincare plane. Indeed putting $s = (1 + i\lambda)/2$ we get

$$S_{\mathrm{t}}(\lambda) = \frac{f(s)}{f(1 - s)} \quad , \tag{216}$$

where

$$f(s) = \frac{\Gamma(s)}{\Gamma(1/2+s)} \tag{217}$$

is a Harrish–Chandra factor for $sl(2,\mathrm{R})$. With this intriguing comment we finish our long and detailed treatment of the $\mathrm{XXX}_{1/2}$ model. From now on I shall describe several directions of development and/or generalization along the similar lines without giving too much details. The first generalization is a XXX model for higher spin.

8 $\mathbf{XXX}_s$ model

Now I consider the spin chain with local spin variables S_n^α realizing the finite dimensional representation of $sl(2)$ in $2s+1$ dimensional space C^{2s+1}, where s is spin, integer or half–integer. I am not ready to write the corresponding hamiltonian. To maintain the integrability I shall find it as a member of the commuting family of operators, generating function for which will be trace of an appropriate monodromy of the family of local Lax operators, satisfying the FCR *a-lá* (44).

The Lax operator $L_{n,a}(\lambda)$ with the auxilialy space $V = \mathrm{C}^2$ does not differ from (32). Indeed, operator, defined in $h_n \otimes V = \mathrm{C}^{2s+1} \otimes \mathrm{C}^2$ by matrix

$$L_{n,a}(\lambda) = \lambda \mathrm{I} + i\sum_\alpha S_n^\alpha \sigma^\alpha = \begin{pmatrix} \lambda + iS_n^3 & iS_n^- \\ iS_n^+ & \lambda - iS_n^3 \end{pmatrix} \tag{218}$$

satisfy the relation

$$R_{a_1,a_2}(\lambda-\mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu) \tag{219}$$

with the same R-matrix $R_{a_1,a_2}(\lambda)$ from (35). The derivation from section 3 is not applicable. We shall not derive (219) here because a more general check will be done below for the XXZ model.

Introducing the monodromy

$$T_a(\lambda) = \prod L_{n,a}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{220}$$

we see, that it satisfies FCR of the form (44), so that its trace

$$F(\lambda) = A(\lambda) + D(\lambda) \tag{221}$$

is a commuting family of operators

$$[F(\lambda), F(\mu)] = 0. \tag{222}$$

This family can be diagonalized by means of Algebraic Bethe Ansatz (ABA). Indeed, we have local vacuum ω_n – the highest weight in C^{2s+1}, the reference state

$$\Omega = \prod \otimes \omega_n \tag{223}$$

with the eigenvalues for $A(\lambda)$, $D(\lambda)$ and $C(\lambda)$

$$A(\lambda)\Omega = \alpha^N(\lambda)\Omega, \quad D(\lambda)\Omega = \delta^N(\lambda)\Omega, \quad C(\lambda)\Omega = 0\ , \tag{224}$$

where

$$\alpha(\lambda) = \lambda + is \ , \quad \delta(\lambda) = \lambda - is \ , \tag{225}$$

and exactly the same exchange relation for $A(\lambda)$, $D(\lambda)$ and $B(\lambda)$ as (67)-(68). Thus the state

$$\Phi(\{\lambda\}) = B(\lambda_1) \dots B(\lambda_l)\Omega \tag{226}$$

is an eigenstate of the family $F(\lambda)$ with the eigenvalue

$$\Lambda(\lambda, \{\lambda\}) = (\lambda + is)^N \prod_{j=1}^{l} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j} + (\lambda - is)^N \prod_{i=1}^{l} \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j} \ , \tag{227}$$

if $\{\lambda\}$ are roots of the BAE

$$\left(\frac{\lambda_k + is}{\lambda_k - is}\right)^N = \prod_{j \neq k}^{l} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i} \quad . \tag{228}$$

When $s = 1/2$ we return to the case already considered above.

However the construction of the local hamiltonian cannot repeat one from section 3. Indeed, in no point λ the Lax operator $L_{n,a}(\lambda)$ is a permutation, just because the quantum space $h_n = \mathrm{C}^{2s+1}$ and auxiliary space $V = \mathrm{C}^2$ are essentially different.

The way out is to find another Lax operator, for which the auxiliary space V is C^{2s+1}.

The existence of such an operator is based on a more general interpretation of the FCR due to V.Drinfeld.

In this interpretation the generating object for Lax operators is a universal R-matrix $\mathcal{R}$ defined as an element in $\mathcal{A} \otimes \mathcal{A}$ for some algebra $\mathcal{A}$, satisfying the abstract Yang–Baxter relation (YBR)

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \ . \tag{229}$$

This equation holds in $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ and rather evident notations are used, i.e.

$$\mathcal{R}_{12} = \mathcal{R} \otimes \mathrm{I}, \quad \mathcal{R}_{23} = \mathrm{I} \otimes \mathcal{R} \quad . \tag{230}$$

The algebra $\mathcal{A}$ must have a family of representations $\rho(\lambda, a)$, parametrized by a discrete label a and continuous parameter λ. For instance the loop algebra of any unitary group has such representations called the evaluation representations. In our case of XXX models the algebra $\mathcal{A}$ was identified by Drinfeld and called Yangian by him. Below on the case of XXZ models we will encounter the q-deformed affine algebra as $\mathcal{A}$.

The concrete Lax operators are obtained via the evaluation representations of the universal $\mathcal{R}$-matrix, i.e.

$$L_{n,a}(\lambda - \mu) = (\rho(a, \lambda) \otimes \rho(n, \mu))\mathcal{R} = R^{a,n}(\lambda - \mu) \quad . \tag{231}$$

The dependence in the LHS on $\lambda - \mu$ reflects some homogeneity in the family of representations $\rho(a, \lambda)$. The Yangian for $sl(2)$ has representations $\rho(a, \lambda)$, where a is just spin label of representations of $sl(2)$, $a = 0, 1/2, 1, \dots$. The relation (36) is obtained if we apply the representation $\rho(1/2,\lambda)\otimes\rho(1/2,\mu)\otimes\rho(s,\sigma)$ to YBR (229), put $\sigma=0$ and identify

$$R_{a_1,a_2}(\lambda)=R^{1/2,1/2}(\lambda)\ ;\quad L_{n,a_1}(\lambda)=R^{1/2,s}(\lambda). \tag{232}$$

However we can use another combination of the representations. Let us rewrite the YBR (229) in the form

$$\mathcal{R}_{12}\mathcal{R}_{32}\mathcal{R}_{31}=\mathcal{R}_{31}\mathcal{R}_{32}\mathcal{R}_{12}\ \ , \tag{233}$$

which can be easily derived from (229) by applying the appropriate permutation to it. Now apply to (233) the representation $\rho(s_1,\lambda)\otimes\rho(s_2,\mu)\otimes\rho(1/2,\sigma)$. We get

$$\begin{aligned}&R^{s_1,s_2}(\lambda-\mu)R^{1/2,s_2}(\sigma-\mu)R^{1/2,s_1}(\sigma-\lambda)=\\ &=R^{1/2,s_1}(\sigma-\lambda)R^{1/2,s_2}(\sigma-\mu)R^{s_1,s_2}(\lambda-\mu)\ \ .\end{aligned} \tag{234}$$

The factors $R^{1/2,s}(\lambda)$ can be identified with the Lax operators $L_{n,a}(\lambda)$ above; the operator $R^{s_1,s_2}(\lambda)$ give us a new Lax operator; we can take representation with spin s_1 as a local quantum space and that with spin s_2 as an auxiliary space. In particular for $s_1=s_2$ we get the Lax operator $L_{n,f}(\lambda)$ we are looking for. Equation (234) is a linear equation, which allows to calculate $L_{n,f}$ if $L_{n,a}$ are known. We shall call $L_{n,f}$ the fundamental Lax operator and label f stems from this.

Let us now calculate $R^{s_1,s_2}(\lambda)$ for representations s_1 and s_2 being the same, using equation (234). To simplify the notation we shall denote two sets of spin variables by S^α and T^α and use the notations

$$L_T(\lambda)=\lambda+i(T,\sigma),\quad L_S(\lambda)=\lambda+i(S,\sigma) \tag{235}$$

for the corresponding Lax operators $R^{1/2,s}$. Here

$$(T,\sigma)=\sum_\alpha T^\alpha\sigma^\alpha \tag{236}$$

and analogously for (S,σ). We shall look for $R^{s_1,s_2}(\lambda)$ in the form

$$R^{s_1,s_2}(\lambda)=\mathrm{P}^{s_1,s_2}r((S,T),\lambda)\ \ , \tag{237}$$

where P^{s_1,s_2} is a permutation in $\mathrm{C}^{2s+1}\otimes\mathrm{C}^{2s+1}$ and (S,T) is a Casimir C

$$C=(S,T)=\sum_\alpha S^\alpha T^\alpha\ \ . \tag{238}$$

Using

$$\mathrm{P}(S,\sigma)\mathrm{P}=(T,\sigma) \tag{239}$$

we rewrite the equation (234) as follows

$$(\lambda-i(T,\sigma))(\mu-i(S,\sigma))r(\lambda-\mu)=r(\lambda-\mu)(\mu-i(T,\sigma))(\lambda-i(S,\sigma))\,. \tag{240}$$

We have due to the property of Pauli matrices σ^α

$$(T,\sigma)(S,\sigma)=(T,S)+i((S\times T),\sigma)\quad , \tag{241}$$

where

$$(S \times T)^{\alpha} = \varepsilon_{\alpha\beta\gamma} S^{\beta} T^{\gamma} \quad . \tag{242}$$

Now using the central property of Casimir we transform equation (240) into

$$(\lambda S^{\alpha} + (T \times S)^{\alpha}) r(\lambda) = r(\lambda)(\lambda T^{\alpha} + (T \times S)^{\alpha}) \ . \tag{243}$$

Due to symmetry, it is enough to consider one out of three equations (243) e.g. the combination

$$(\lambda S^{+} + i(T^{3} S^{+} - S^{3} T^{+})) r(\lambda) = r(\lambda)(\lambda T^{+} + i(T^{3} S^{+} - S^{3} T^{+})) \ . \tag{244}$$

We shall use a convenient variable J instead of Casimir (S, T); taking into account that representations for S and T are irreducible we have

$$(S + T)^{2} = S^{2} + T^{2} + 2(S, T) = 2s(s + 1) + 2(S, T) = J(J + 1) \ , \tag{245}$$

where the operator J have an eigenvalue j in each irreducible representation D_j entering the Klebsch–Gordan decomposition

$$D_s \otimes D_s = \sum_{j=0}^{2s} D_j \quad . \tag{246}$$

We shall look for operator $r(\lambda)$ as a function of J. To find it we shall use the equation (244) in the subspace of the highest weights in each D_j, i.e. put

$$T^{+} + S^{+} = 0 \quad . \tag{247}$$

This is permissible because

$$[T^{+} S^{3} - S^{+} T^{3} \ , \ T^{+} + S^{+}] = 0 \quad . \tag{248}$$

In this subspace due to the fact, that in general

$$(S + T)^{2} = (S^{3} + T^{3})^{2} + S^{3} + T^{3} + (S^{-} + T^{-})(S^{+} + T^{+}) \tag{249}$$

we can identify

$$J = S^{3} + T^{3} \quad . \tag{250}$$

In the constrainted subspace the equation (244) reduces to

$$(\lambda S^{+} + iJS^{+}) r(\lambda, J) = r(\lambda, J)(-\lambda S^{+} + iJS^{+}) \tag{251}$$

Now we use the commutation relation

$$S^{+} J = S^{+}(S^{3} + T^{3}) = (S^{3} + T^{3} - 1) S^{+} = (J - 1) S^{+} \tag{252}$$

to turn (251) into the functional equation

$$(\lambda + iJ) r(\lambda, J - 1) = r(\lambda, J)(-\lambda + iJ) \tag{253}$$

with solution

$$r(J, \lambda) = \frac{\Gamma(J + 1 + i\lambda)}{\Gamma(J + 1 - i\lambda)} \quad , \tag{254}$$

normalized in such a way

$$r(J,0) = I; \quad r(J,-\lambda)r(J,\lambda) = \mathrm{I} \ . \tag{255}$$

Of course in our case of finite dimensional D_s we are interested only in $r(J,\lambda)$ for J taking values $J = 0, 1, 2, \ldots, 2s$. But in what follows we shall use the representations with any complex s and then formula (254) will be used in all generality.

Having this Lax operator $L_{n,f}(\lambda)$ we can write one more variant of FCR

$$R_{f_1,f_2}(\lambda-\mu)L_{n,f_1}(\lambda)L_{n,f_2}(\mu) = L_{n,f_2}(\mu)L_{n,f_1}(\lambda)R_{f_1,f_2}(\lambda-\mu) \ . \tag{256}$$

From this we infer, that the spectral invariants of the monodromy

$$T_f(\lambda) = L_{N,f}(\lambda)L_{N-1,f}(\lambda)\ldots L_{1,f}(\lambda) \tag{257}$$

(i.e. $F_f(\lambda) = \mathrm{tr}_f T_f(\lambda)$) are commuting

$$[F_f(\lambda), F_f(\mu)] = 0 \ . \tag{258}$$

The relation (234) with $s_1 = s_2$, where $R^{1/2,s}(\lambda-\mu)$ is used as R-matrix and $R^{1/2,s}(\mu)$ and $R^{s,s}(\lambda)$ as Lax operators leads to the commutativity of the families $F_f(\lambda)$ and $F_a(\lambda)$

$$[F_f(\lambda), F_a(\mu)] = 0 \ . \tag{259}$$

Thus we can use $F_f(\lambda)$ to get observables and $F_a(\lambda)$ to construct BAE.

Repeating the considerations in section 3 we get

$$F_f(0) = U = e^{iP} \tag{260}$$

and

$$H = i\,\frac{d}{d\lambda}\ln F_f(\lambda)\bigg|_{\lambda=0} = \sum_{n=1}^{N} H_{n,n+1} \quad , \tag{261}$$

where

$$H_{n,n+1} = i\,\frac{d}{d\lambda}\ln r(J,\lambda)\bigg|_{\lambda=0} \quad , \tag{262}$$

where J is constructed via local spins S_n^α and S_{n+1}^α as

$$J(J+1) = 2\sum_\alpha (S_n^\alpha S_{n+1}^\alpha) + 2s(s+1) \tag{263}$$

From (254) and (262) we get

$$H_{n,n+1} = -2\psi(J+1) \quad , \tag{264}$$

where $\psi(z)$ is logarithmic derivative of $\Gamma(z)$. For positive integer n we have

$$\psi(1+n) = \sum_{k=1}^{n}\frac{1}{k} - \gamma \ , \tag{265}$$

where γ is the Euler constant, and it allows to express $H_{n,n+1}$ as a polinomial in invariant $\sum_\alpha S_n^\alpha S_{n+1}^\alpha = C_{n,n+1}$ as follows

$$H_{n,n+1} = \sum_{j=1}^{2s} c_k (C_{n,n+1})^k = f_{2s}(C_{n,n+1}) \ , \tag{266}$$

where the polinomial $f_{2s}(x)$ can be written using Lagrange interpolation as

$$f_{2s}(x) = \sum_{j=1}^{2s} \left(\sum_{k=1}^{j} \frac{1}{k} - \gamma \right) \prod_{l=0}^{2s} \frac{x - x_l}{x_j - x_l}, \quad x_l = \frac{1}{2}(l(l+1) - 2s(s+1)) \ . \tag{267}$$

In particular for $s = 1$ we have

$$c_1 \ = \ -c_2 \ \ , \tag{268}$$

so that the hamiltonian

$$H = \sum_{\alpha,n} \left(S_n^\alpha S_{n+1}^\alpha - (S_n^\alpha S_{n+1}^\alpha)^2 \right) \tag{269}$$

is integrable for the representation of local spins in C^3. The naive generalization of the hamiltonian (27) by simple substitution of operators of spins 1/2 by those of spin 1 is not integrable.

The construction of the integrable hamiltonians for spin s magnetic chains is one of real achievements of the Algebraic Bethe Ansatz. Indeed, without well understood connection of integrable hamiltonians and Lax operators there is no hope to reproduce the formulas (266), (267).

Now I shall give without derivation the expression for the eigenvalue $\Lambda_f(\lambda, \{\lambda\})$ of family $F_f(\lambda)$ on the Bethe vector $\Phi(\{\lambda\})$:

$$\Lambda_f(\lambda, \{\lambda\}) = \sum_{m=-s}^{s} \alpha_m(\lambda)^N \prod_{k=1}^{l} c_m(\lambda - \lambda_k) \ \ . \tag{270}$$

The complete list of $\alpha_m(\lambda)$ and $c_m(\lambda)$ is not important. Suffice to say, that

$$\alpha_m(0) \ = \ 0 \tag{271}$$

for all $m = -s, \ -s+1, \ldots s-1$ and $\alpha_s(0) = 1$. Further,

$$c_s(\lambda) = \frac{\lambda - is}{\lambda + is} \ \ . \tag{272}$$

The derivation is based on the relation (234) which allows to commute the diagonal elements of matrix $T_f(\lambda)$ with $B(\lambda)$ from $T_a(\lambda)$. From (270) we read the momentum and energy of quasiparticles

$$p(\lambda) \ = \ \frac{1}{i} \ln \frac{\lambda + is}{\lambda - is} \ \ ; \tag{273}$$

$$\epsilon(\lambda) \ = \ -\frac{s}{\lambda^2 + s^2} \ \ . \tag{274}$$

The relation

$$\epsilon(\lambda) = \frac{1}{2} \frac{d}{d\lambda} p(\lambda) \tag{275}$$

holds; also it is $p(\lambda)$ which enters the LHS of BAE (228), so we do not need the Lax operator $L_{n,f}(\lambda)$ to calculate $p(\lambda)$ and $\epsilon(\lambda)$.

The BAE (228) can be investigated similarly to what was done in case $s = 1/2$. In the limit $N \to \infty$ the roots are combined into complexes of type M. The momenta of these complexes are distinct of those for spin 1/2. The S-matrices are defined by the RHS of BAE and are the same for any spin. One can make the counting of roots and get completeness by showing, that full number of states (for which Bethe vectors are the highest weights) is equal to $(2s+1)^N$.

Finally, we add some comments on the thermodynamic limit in the antiferromagnetic case. The ground state Φ_0 is given by $\nu_{s-1/2} = N/2$, $\nu_M = 0$, $M \neq s-1/2$. The excitations correspond to $\nu_{s-1/2}$ macroscopic and all other ν_M finite. They have particle interpretation as spin 1/2 particles with the same one particle momentum and energy as in the case $s = 1/2$. However the counting of their states shows, that excitations have more degrees of freedom, than just rapidity λ and spin ε.

I shall describe the picture of excitations (without derivation) using the language of the creation operators. In addition to spin label ε and rapidity λ this operator is supplied by a pair of indeces a, a' assuming integer values from 0 to $2s$ and subject to condition $a' = a \pm 1$. The n-particle excitation is given by a "string"

$$Z^{a_0,a_1}_{\varepsilon_1}(\lambda_1) Z^{a_1,a_2}_{\varepsilon_2}(\lambda_2) \dots Z^{a_{n-1},a_n}_{\varepsilon_n}(\lambda_n)\Phi_0 \ , \tag{276}$$

where $a_0 = 0$ and $a_n = 0$. The counting of Bethe vectors is in exact accord with this picture.

The exchange relation for operators Z employs the S-matrix, which is a tensor product of the spin 1/2 S-matrix from section 7 and S-matrix, which acts on the indeces a.

The latter will be denoted by $S\left(\lambda \left| \begin{matrix} & b & \\ a & & d \\ & c & \end{matrix} \right.\right)$ and it enters the exchange relations as follows

$$Z^{ab}(\lambda) Z^{bc}(\mu) = \sum_d Z^{ad}(\mu) Z^{dc}(\lambda) S\left(\lambda-\mu \left| \begin{matrix} & b & \\ a & & c \\ & d & \end{matrix} \right.\right) \ , \tag{277}$$

where we suppressed the usual spin variables. They are easily introduced if we write the full S-matrix as

$$S = S^{1/2,1/2}(\lambda) \otimes S\left(\lambda \left| \begin{matrix} & b & \\ a & & d \\ & c & \end{matrix} \right.\right) \ . \tag{278}$$

The consistency of relations of type (277) is based on a star-triangle relation for $S\left(\lambda \left| \begin{matrix} & b & \\ a & & d \\ & c & \end{matrix} \right.\right)$

$$\sum_p S\left(\lambda-\mu \left| \begin{matrix} & b & \\ a & & c \\ & p & \end{matrix} \right.\right) S\left(\lambda-\sigma \left| \begin{matrix} & c & \\ p & & d \\ & e & \end{matrix} \right.\right) S\left(\mu-\sigma \left| \begin{matrix} & p & \\ a & & e \\ & f & \end{matrix} \right.\right) =$$
$$= \sum_p S\left(\mu-\sigma \left| \begin{matrix} & c & \\ b & & d \\ & p & \end{matrix} \right.\right) S\left(\lambda-\sigma \left| \begin{matrix} & b & \\ a & & p \\ & f & \end{matrix} \right.\right) S\left(\lambda-\mu \left| \begin{matrix} & p & \\ f & & d \\ & e & \end{matrix} \right.\right) .$$

The explicit expression for $S\left(\lambda \left| \begin{matrix} & b & \\ a & & d \\ & c & \end{matrix} \right.\right)$ contains factor $S_0(\lambda)$ given by

$$S_0(\lambda) = \exp\left\{ -i \int_0^\infty \frac{dx}{x} \frac{\sin(\lambda x)\sinh(s+1/2)x}{\cosh(x/2)\sinh(s+1)x} \right\} \ . \tag{279}$$

In particular

$$S\left(\lambda \left| a \begin{array}{cc} a+1 & \\ & a+2 \\ a+1 & \end{array}\right.\right) = S\left(\lambda \left| a \begin{array}{cc} a-1 & \\ & a-2 \\ a-1 & \end{array}\right.\right) = S_0(\lambda)\ ; \tag{280}$$

and

$$S\left(\lambda \left| a \begin{array}{cc} a+1 & \\ & a \\ a+1 & \end{array}\right.\right) = \frac{\sinh\left(\frac{\pi}{2s+2}(\lambda + i(a+1))\right)\sin\frac{\pi}{2s+2}}{\sin\left(\frac{\pi}{2s+2}(a+1)\right)\sinh\frac{\pi}{2s+2}(\lambda - i)} S_0(\lambda)\ . \tag{281}$$

Other components

$$S\left(\lambda \left| a \begin{array}{cc} a-1 & \\ & a \\ a-1 & \end{array}\right.\right),\quad S\left(\lambda \left| a \begin{array}{cc} a-1 & \\ & a \\ a+1 & \end{array}\right.\right),\quad S\left(\lambda \left| a \begin{array}{cc} a+1 & \\ & a \\ a-1 & \end{array}\right.\right) \tag{282}$$

contain similar trigonometric factor besides $S_0(\lambda)$.

Thus the antiferromagnetic spin s chain has very remarkable excitations. They are particles with rapidity and spin 1/2, but also kinks, relating the "local vacua", labeled by $a = 0, \ldots, 2s$. Only transitions between the adjacent vacua are allowed in the physical Hilbert space. Some analogy with the Landau–Ginsburg picture in the topological field theory is evident, but not explored yet. On this intriguing note I finish the general discussion of the XXX_s model.

9 XXX_s spin chain. Applications to the physical systems

The appearance of a parameter s in our disposal allows to use it to construct some models, going beyond the usual spin chains. Of particular interest is the limit of infinite spin $s \to \infty$, combined with the formal continuous limit $\Delta \to 0$, which can be realized in many variants. I shall show, that such representative models of quantum field theory as Nonlinear Schroedinger equation (NLS) and S^2 nonlinear σ-model can be obtained from the XXX_s chain in this limit. The first model is nonrelativistic and of prime interest in the condensed matter physics, but the second one is interesting model of relativistic field theory.

I begin with the NLS model, and employ the realization of spin variables via the complex canonical variables from section 1:

$$S_n^+ = \chi_n^*(2s - \chi_n^*\chi_n)\ ;\quad S_n^- = \chi_n\ ;\quad S_n^3 = \chi_n^*\chi_n - s \tag{283}$$

and suppose, that s is some complex number.

The canonical commutation relations for χ_n^*, χ_n assume the usual form

$$[\chi_m, \chi_n^*] = \delta_{mn}\ . \tag{284}$$

If Δ is a lattice spacing, χ_n and χ_n^* have order $\Delta^{1/2}$ in all expressions, where they enter in the normal order, i.e. all χ_n^* to the left of all χ_m with the same n.

The invariant $C_{n,n+1}$ of two adjacent spins S_n and S_{n+1} is given by

$$\begin{aligned} 2C_{n,n+1} &= 2\sum_\alpha S_n^\alpha S_{n+1}^\alpha = 2S_n^3 S_{n+1}^3 + S_n^- S_{n+1}^+ + S_n^+ S_{n+1}^- = \\ &= 2s^2 - 2s(\chi_n^* - \chi_{n+1}^*)(\chi_n - \chi_{n+1}) - (\chi_n^* - \chi_{n+1}^*)^2 \chi_{n+1}\chi_n. \end{aligned} \tag{285}$$

The second term in the RHS looks satisfactory, in the formal continuous limit it leads to the quadratic form of derivatives of field $\chi(x), \chi^*(x)$

$$\chi_n = \Delta^{1/2}\chi(x), \quad \chi_n^* = \Delta^{1/2}\chi^*(x), \quad x = n\Delta \tag{286}$$

due to the prescription

$$\chi_{n+1} = \Delta^{1/2}(\chi(x) + \Delta\chi'(x) + O(\Delta^2)) \quad . \tag{287}$$

However the last term looks bad and does not lead to the desired expressions $(\chi^*(x))^2(\chi(x))^2$ characteristic of the NLS model. The remedy is to use equivalent variables ψ_n^*, ψ_n on the lattice

$$\psi_n = (-1)^n\chi_n, \quad \psi_n^* = (-1)^n\chi_n^* \tag{288}$$

and consider ψ_n^*, ψ_n as producing the field $\psi(x), \psi^*(x)$ in the continuous limit as in (286). In the new variables we have

$$2C_{n,n+1} = 2s^2 - 2s(\psi_n^* + \psi_{n+1}^*)(\psi_n + \psi_{n+1}) + (\psi_n^* + \psi_{n+1}^*)^2\psi_{n+1}\psi_n. \tag{289}$$

Now the second term in the RHS is bad, but we add and substruct $4s(\psi_n^*\psi_n + \psi_{n+1}^*\psi_{n+1})$ to change it into

$$\begin{aligned} 2C_{n,n+1} &= 2s^2 - 4s(\psi_n^*\psi_n + \psi_{n+1}^*\psi_{n+1}) + \\ &+ 2s(\psi_{n+1}^* - \psi_n^*)(\psi_{n+1} - \psi_n) + (\psi_n^* + \psi_{n+1}^*)^2\psi_{n+1}\psi_n \quad , \end{aligned} \tag{290}$$

which in the continuous limit will contain only good densities $n(x) = \psi^*\psi$, $h_0(x) = \psi'^*\psi'$ and $h_1(x) = (\psi^*)^2(\psi)^2$. We introduce the operator J via

$$\begin{aligned} J(J+1) &= 2s(s+1) + 2C_{n,n+1} = \\ &= 4s^2 + 2s - 8s\Delta n(x) + 2s\Delta^3 h_0(x) + 4\Delta^2 h_1(x) \end{aligned} \tag{291}$$

and consider the limit $s \to \infty$, $\Delta \to 0$, $s\Delta = g$, where g is a new fixed parameter. We see, that the relation (291) allows to obtain the asymptotics of J in this limit

$$J = 2s + \frac{a}{s} + \frac{b}{s^2} + \dots \quad . \tag{292}$$

Substituting this into the expression

$$H_{n,n+1}(J) = 2\,\frac{d}{dz}\ln\Gamma(z)\Big|_{z=1+J} \tag{293}$$

from section 8, and using Stirling formula for the asymptotics of $\Gamma(z)$ and formal rule

$$\Delta\sum_n = \int dx \tag{294}$$

we get for hamiltonian the expression

$$H_{\text{lattice}} = \text{const} - \frac{1}{s}N + \frac{g^2}{s^3}H^{\text{NLS}} + \dots \quad , \tag{295}$$

where

$$N = \int \psi^* \psi dx \tag{296}$$

$$H^{\text{NLS}} = \int \left[|\psi'(x)|^2 + g(\psi^*)^2(\psi)^2 \right] dx \quad . \tag{297}$$

This agrees nicely with the dispersion law for the energy of quasiparticles

$$h(\lambda) = \frac{s}{\lambda^2 + s^2} = \frac{1}{s} - \frac{\lambda^2}{s^3} + \dots \quad , \tag{298}$$

if we assume, that the momentum p of NLS particle is connected with the rapidity of XXX particle by a simple scale

$$p = \lambda / g \quad . \tag{299}$$

The state Ω plays the role of no particle state of the NLS model. The bound states survive in the limit $s \to \infty$ when $g < 0$. For $g > 0$ the physical hamiltonian is usually taken in the form $H - \mu N$, where μ is a chemical potential. Here the problem of ground state reappears and Dirac sea of particles, created by $\psi^*(x)$ from Ω is to be used. The analogous problem for the original magnetic chain consists in the inclusion of magnetic field into hamiltonian

$$H \to H - \mu S^3 \quad . \tag{300}$$

This changes the picture of Dirac sea: only quasiparticles with rapidities, confined to some finite interval $-B \le \lambda \le B$ form the Dirac sea. The excitations are holes and quasiparticles with $|\lambda| \ge B$. I shall not treat this case in any detail and finish the discussion of the NLS model.

I turn to the second example — the nonlinear σ-model with the field $n(x)$ taking values in the 2-sphere S^2.

The classical field $n(x)$ can be described as a vector $n = (n_1, n_2, n_3)$ subject to constraint

$$(n, n) = n_1^2 + n_1^2 + n_3^2 = 1 \tag{301}$$

for all x. The canonical conjugate variable may be taken in the form

$$l = \frac{1}{\gamma} \partial_0 n \times n, \quad (l, n) = 0. \tag{302}$$

The canonical Poisson brackets

$$\{l^\alpha(x), l^\beta(y)\} = \varepsilon^{\alpha\beta\gamma} l^\gamma(x) \delta(x - y) \ ; \tag{303}$$

$$\{l^\alpha(x), n^\beta(y)\} = \varepsilon^{\alpha\beta\gamma} n^\gamma(x) \delta(x - y) \ ; \tag{304}$$

$$\{n^\alpha(x), n^\beta(y)\} = 0 \tag{305}$$

and hamiltonian

$$H = \frac{\gamma}{2} \int \left(l^2 + \frac{n'^2}{\gamma^2} \right) dx \tag{306}$$

follow from the lagrangian

$$\mathcal{L} = \frac{1}{2\gamma} \partial_\mu n \, \partial_\mu n \tag{307}$$

and constraint (301) in a usual way. Here γ is a coupling constant, which is relevant only in quantum case.

If we regularize the model going to the chain and introducing the variables n_k, l_k as follows,

$$l_k = \Delta l(x), \quad n_k = n(x), \quad x = k\Delta \ , \tag{308}$$

the brackets (303)–(305) will assume the ultralocal form

$$\{l_m^\alpha, l_n^\beta\} = \varepsilon^{\alpha\beta\gamma} l_m^\gamma \delta_{mn} \ ; \tag{309}$$
$$\{l_m^\alpha, n_n^\beta\} = \varepsilon^{\alpha\beta\gamma} n_m^\gamma \delta_{mn} \ ; \tag{310}$$
$$\{n_m^\alpha, n_n^\beta\} = 0 \quad . \tag{311}$$

For each lattice site m the phase space is an orbit of the group $E(3)$ of motions of R^3 corresponding to the choice of Casimirs

$$n^2 = 1; \quad (n, l) = 0 \ , \tag{312}$$

which is also cotangent bundle of S^2. Corresponding quantum Hilbert space can be realized as $L_2(\mathrm{S}^2)$.

To make contact with the spin chains I mention, that this Hilbert space can be realized as an infinite spin limit of a Hilbert space of a pair of spin variables of spin s. Indeed, comparing the Klebsch–Gordan decomposition, already mentioned above,

$$D_s \otimes D_s = \sum_{j=0}^{2s} D_j \tag{313}$$

and decomposition of $L_2(\mathrm{S}^2)$

$$L_2(\mathrm{S}^2) = \sum_{j=0}^{\infty} D_j \ , \tag{314}$$

we see that

$$L_2(\mathrm{S}^2) = \lim_{s\to\infty} D_s \otimes D_s \ . \tag{315}$$

Thus the σ-model variables n_k, l_k must be realized through the pair of spin variables S_{2k-1}, S_{2k}. The most naive way

$$l_k = S_{2k-1} + S_{2k} \ ; \tag{316}$$
$$n_k = \frac{1}{2s}(S_{2k} - S_{2k-1}) \tag{317}$$

or inversely

$$S_{2k-1} = \frac{1}{2} l_k - s n_k \ ; \tag{318}$$
$$S_{2k} = \frac{1}{2} l_k + s n_k \tag{319}$$

works. Indeed, from the spin commutation relations we reproduce the first two relations (309), (310) (in their quantum form) exactly and have

$$[n_m^\alpha, n_n^\beta] = \frac{i}{4s^2} \varepsilon^{\alpha\beta\gamma} l^\gamma \delta_{m,n} \tag{320}$$

with RHS vanishing for $s \to \infty$. Furthermore we have

$$(l_k, n_k) = 0 \; ; \quad 4s^2 n_k^2 + l_k^2 = 4s(s+1) \quad , \tag{321}$$

which reproduce (312) in the limit $s \to \infty$.

Now the idea is evident: we are to consider the XXX_s chain on the lattice of even length and take two adjacent points $(2k-1, 2k)$ as one lattice point for σ-model. Let us see, how the hamiltonian (306) appears in the formal continuous limit. For that we are to estimate the invariants $C_{2k-1,2k}$ and $C_{2k,2k+1}$. We have

$$C_{2k-1,2k} = \frac{1}{4} l_k^2 - s^2 n_k^2 = \frac{1}{2} l_k^2 - s(s+1) \tag{322}$$

and

$$C_{2k,2k+1} = \frac{1}{4} l_k l_{k+1} + \frac{s}{2}(n_k l_{k+1} - n_{k+1} l_k) - s^2 n_k n_{k+1} \tag{323}$$

Now we estimate the invariant in Δ expansion using the convention (308) and

$$n_{k+1} = n(x) + \Delta n'(x) + \frac{1}{2}\Delta^2 n''(x) + \ldots \tag{324}$$

to get

$$\begin{aligned} 2C_{2k-1,2k} + 2s(s+1) &= \Delta^2 l(x)^2 \quad ; \\ 2C_{2k,2k+1} + 2s(s+1) &= \Delta^2 \left(l(x)^2 - 2s(n'(x), l(x)) - s^2(n''(x), n(x)) \right) . \end{aligned} \tag{325}$$

Corresponding operators $J_{2k-1,2k}$ and $J_{2k,2k+1}$ have order Δ^2 so that

$$H_{2k-1,2k} + H_{2k,2k+1} = 2\psi(1 + J_{2k-1,2k}) + 2\psi(1 + J_{2k,2k+1)}) \tag{326}$$

can be easily calculated. Taking the sum over k with our usual understanding

$$\Delta \sum_k = \int dx \tag{327}$$

and integrating by parts we get for the hamiltonian H^σ

$$H^\sigma = \frac{1}{2s\Delta\psi'(1)} H^{\mathrm{XXX}} = \frac{1}{s} \int \left(\left(l - \frac{1}{2} s n' \right)^2 + \frac{s^2}{4} (n')^2 \right) dx \; , \tag{328}$$

which turns into (306) if we comment, that the map

$$l \to l + \alpha n' \; , \quad n \to n \tag{329}$$

for any α is a canonical transformator for the brackets (303)–(305). The coupling constant γ is connected with spin s as follows

$$\gamma = \frac{2}{s} \quad . \tag{330}$$

The shift (329) is produced by a topological term, added to the lagrangian (307),

$$\mathcal{L}_\vartheta = \frac{\vartheta}{8\pi} (\partial_\mu n \times \partial_\nu n, n) \varepsilon_{\mu\nu} \; . \tag{331}$$

Indeed, such addition changes only the definition of the canonical momenta

$$l \to l + \frac{\vartheta}{4\pi} n' \tag{332}$$

and thus we can interpret the model obtained from XXX chain as nonlinear σ-model with ϑ-term with

$$\vartheta = 2\pi s \quad . \tag{333}$$

The ϑ-term with $\vartheta = 2\pi n$ is trivial; in other words ϑ is defined mod 2π. Thus we see an important difference of integer and half–integer spin s used in our construction. The topological term is present only if spin is half–integer. This phenomenon and its consequences are discussed in detail by Professor I. Affleck in his Les–Houches lectures of 1993 and I can only refer you to the corresponding proceedings.

Unfortunately the described connection was not yet realized in any real computation for the nonlinear σ-model. In particular we expect, that the magnetic chain in the relevant thermodynamic limit must have an excitation of spin 1 with relativistic dispersion law

$$p(\lambda) = c \sinh \lambda \quad ; \tag{334}$$

$$\epsilon(\lambda) = c \cosh \lambda \tag{335}$$

with parameter c being exponentially small for $s \to \infty$

$$c = e^{-s/2} \quad . \tag{336}$$

This will lead to the realizations of dimensional transmutation program giving mass m via the lattice spacing Δ and the coupling constant $\gamma = 2/s$

$$m = \frac{1}{\Delta} e^{-1/\gamma} \quad . \tag{337}$$

Apparently to achieve this we are to understand which portion of the infinite tensor product of the spin chain is compatible with the formal continuous limit we just described.

One comment could be relevant to this program. It is natural to combine the adjacent Lax operator into product $L_{2k,a}(\lambda)L_{2k-1,a}(\lambda)$ and perform the change of variables (318), (319) there. It turns out, that this new Lax operator has a new local vacuum in $L_2(\mathrm{S}^2)$ and the corresponding BAE look like

$$\left(\frac{\lambda_k - is}{\lambda_k + is}\,\frac{\lambda_k + i(s+1)}{\lambda_k - i(s+1)}\right)^{N/2} = \prod_{j \neq k} \frac{\lambda_k - \lambda_j - i}{\lambda_k - \lambda_j + i} \quad . \tag{338}$$

The alternating value of spin necessary to maintain our conventions (308) is manifest here. However the proper choice of solutions to these equations is not done yet. I leave it as a challenge and stop here the discussion of nonlinear σ-model.

I also finish considerations of XXX chains (with one revisit in section 12) and turn to the XXZ chains.

10 XXZ model

As was told above, this model is a deformation of XXX model with one new parameter. We shall denote this parameter by q or γ with $q = e^{i\gamma}$. The deformation uses the q-analogues of usual number

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sin \gamma x}{\sin \gamma} = \prod_{n=-\infty}^{\infty} \left(\frac{x + n\pi\gamma^{-1}}{1 + n\pi\gamma^{-1}} \right) \quad , \tag{339}$$

which effectively change the complex plane to a strip via the multiplicative averaging in (339). We begin with the construction of the Lax operator of XXZ model using the matrix analogue of this averaging. For classical spin variables the formal averaging

$$L_{n,a}^{\mathrm{XXZ}}(\lambda) = \prod_{k=-\infty}^{\infty} L_{n,a}^{\mathrm{XXX}}(\lambda + ik\pi\gamma^{-1}) \tag{340}$$

can be evaluated to lead to the expression

$$L_{n,a}^{\mathrm{XXZ}}(\lambda) = \frac{1}{\sin \gamma} \begin{pmatrix} \sinh \gamma(\lambda + i\tilde{S}_n^3) & \sin \gamma \tilde{S}_n^- \\ \sin \gamma \tilde{S}_n^+ & \sinh \gamma(\lambda - i\tilde{S}_n^3) \end{pmatrix} \quad , \tag{341}$$

where $\tilde{S}_n^3$, $\tilde{S}_n^{\pm}$ are some function of the original spin variables S_n^3, $S_n^{\pm}$ of XXX model. Taking this as heuristic consideration we shall look for the Lax operator $L_{n,a}(\lambda)$ in the form

$$L_{n,a}(x) = \begin{pmatrix} xq^{S_n^3} - x^{-1}q^{-S_n^3} & (q - q^{-1})S_n^- \\ (q - q^{-1})S_n^+ & xq^{-S_n^3} - x^{-1}q^{S_n^3} \end{pmatrix} \quad , \tag{342}$$

using multiplicative spectral parameter

$$x = q^{-i\lambda} \tag{343}$$

and quantum operator $q^{S_n^3}$, $S_n^{\pm}$. We shall check, that $L_{n,a}(x)$ satisfy the FCR

$$R_{a_1,a_2}(x/y)\, L_{n,a_1}(x) L_{n,a_2}(y) = L_{n,a_2}(y) L_{n,a_1}(x) R_{a_1,a_2}(x/y) \quad , \tag{344}$$

where R is a q-deformed analogue of the R-matrix from section 4 (see (73))

$$R = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix} \tag{345}$$

with

$$a = qx - q^{-1}x^{-1}; \quad b = x - x^{-1}; \quad c = q - q^{-1}. \tag{346}$$

To check FCR (344) it is convenient to twist it a little, introducing

$$\tilde{L}_{n,a}(x) = Q(x) L_{n,a}(x) Q^{-1}(x) \quad , \tag{347}$$

$$\tilde{R}_{a_1,a_2}(x) = Q(x) \otimes Q(y)\; R_{a_1,a_2}(x/y)\; Q^{-1}(x) \otimes Q^{-1}(y) \; , \tag{348}$$

where $Q(x)$ is a matrix in the auxiliary space

$$Q(x) = \begin{pmatrix} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix} . \tag{349}$$

It is clear, that FCR is true for original $L_{n,a}(x)$ if it holds for $\tilde{L}_{n,a}(x)$.

Now observe, that $\tilde{L}_{n,a}(x)$ and $\tilde{R}_{a_1,a_2}(x)$ have a simple x-dependence

$$L = xL_+ - x^{-1}L_- \ , \quad R = xR_+ - x^{-1}R_- \ , \tag{350}$$

where we dropped indeces n, a_1, a_2, $\tilde{}$, and matrices L_+, L_-, R_+, R_- are given by

$$L_+ = \begin{pmatrix} q^{S^3} & (q-q^{-1})S^- \\ 0 & q^{-S^3} \end{pmatrix}; \quad L_- = \begin{pmatrix} q^{-S^3} & 0 \\ -(q-q^{-1})S^+ & q^{S^3} \end{pmatrix} \quad ; \tag{351}$$

$$R_+ = \begin{pmatrix} q & & & \\ & 1 & q-q^{-1} & \\ & & 1 & \\ & & & q \end{pmatrix} \quad ; \tag{352}$$

$$R_- = \begin{pmatrix} q^{-1} & & & \\ & 1 & 0 & \\ & -(q-q^{-1}) & 1 & \\ & & & q^{-1} \end{pmatrix} \quad . \tag{353}$$

It is clear, that

$$R_- = \mathrm{P} R_+^{-1} \mathrm{P} \tag{354}$$

and

$$R_+ - R_- = \left(q - q^{-1}\right) \mathrm{P} \ , \tag{355}$$

where P is a permutation matrix.

Separation of spectral parameters in FCR shows, that it is true, if the following 7 relations hold:

$$R L^1_{\pm} L^2_{\pm} = L^2_{\pm} L^1_{\pm} R \ , \tag{356}$$

where R can be R_+ and R_- and labels 1 and 2 substitute a_1 and a_2; furthermore

$$R_+ L^1_+ L^2_- = L^2_- L^1_+ R_+ \ ; \tag{357}$$

$$R_- L^1_- L^2_+ = L^2_+ L^1_- R_- \tag{358}$$

and

$$R_+ L^1_- L^2_+ - R_- L^1_+ L^2_- = L^2_+ L^1_- R_+ - L^2_- L^1_+ R_- . \tag{359}$$

Only three of them are independent and we take as such two of the relation (356) for L_+ and L_- separately and relation (357). The two other relations in (356) and relation (358) are easily reduced to the chosen ones if one applies the permutation taking into account, that

$$L^1 = \mathrm{P} L^2 \mathrm{P} \ . \tag{360}$$

Finally relation (359) is checked if one uses the property (355) to substitute unwanted R_+ by R_- and vice versa.

Now it is easy to check, that the basic relations, say

$$R_+ L^1_+ L^2_+ = L^2_+ L^1_+ R_+ \ ; \tag{361}$$

$$R_+ L^1_- L^2_- = L^2_- L^1_- R_+ \ ; \tag{362}$$

$$R_+ L^1_+ L^2_- = L^2_- L^1_+ R_+ \tag{363}$$

are satisfied, if q^{S^3}, S^+ and S^- satisfy the commutation relations of the q-deformed $sl(2)$ from section 2. This finishes the proof of the FCR (344). It is worth to mention, that the XXX_s Lax operator (218) is obtained from (342) in the limit $q \to 1$, so FCR is proved also for it.

Another comment is, that we can consider the relations (361)–(363) together with the structure (351) as alternative definition of the q-deformed $sl_q(2)$ algebra, which will prove to be convenient in what follows.

If we renormalize $R_\pm$ by

$$R_+ \to q^{-1/2} R_+ , \quad R_- \to q^{-1/2} R_- \tag{364}$$

we see that they take form (351) where q^{S^3} and $S^\pm$ realize the 2-dimensional representation

$$q^{S^3} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad S^+ = \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix}, \quad S^- = \begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{pmatrix}, \tag{365}$$

deforming the Pauli matrices.

The R-matrix and Lax operators above can be considered as representation of the universal $\mathcal{R}$-matrix, which in this case corresponds to the q-deformed affine algebra $\widehat{sl_q}(2)$.

In some sense this algebra (see a lot about it in Professor Miwa lectures) is more natural object than Yangian, which is a special contraction of it when $q \to 1$ (or γ tends to 0) in conjunction with a renormalization of the grading parameter λ.

The q-deformed $sl(2)$ has finite dimensional representations, analogous to those in the nondeformed case; their dimension is $2s+1$, where spin s is integer or half–integer. Besides it has other interesting representations, called cyclic and already briefly mentioned in section 2; these representations have no limit, when $q \to 1$.

The deformed representations in C^{2s+1} are of highest weight type, so that there is a state ω such, that

$$S^+ \omega = 0 \, , \quad q^{S^3} \omega = q^s \omega \; . \tag{366}$$

This allows to repeat all constructions of Algebraic Bethe Anzatz and the BAE assume the form

$$\frac{\sinh(\lambda_k + is\gamma)}{\sinh(\lambda_k - is\gamma)} = \prod_{j \neq k} \frac{\sinh(\lambda_k - \lambda_j + i\gamma)}{\sinh(\lambda_k - \lambda_j - i\gamma)} \quad , \tag{367}$$

where we returned to the additive spectral parameter λ to make (367) look rather similar to BAE (228).

The investigation of these equations is to large extent similar to what was done before in XXX case. One encounters complexes of roots, one can estimate the number of Bethe vectors etc. A new feature is a role, played by the arithmetic nature of γ. For instance if γ/π is rational then q is a root of unity and all specifics of $sl_q(2)$ for such a case is to be taken into account. We have no time to discuss it in any detail.

I turn now to the problem of constructing the local hamiltonian. For that we need to control better the invariant of the pair of spins.

It is convenient to use the matrix

$$L = L_+ L_-^{-1} \tag{368}$$

to describe the deformed spins. The matrix L_-^{-1} is given by

$$L_-^{-1} = \begin{pmatrix} q^{S^3} & 0 \\ (q - q^{-1})S^+ & q^{-S^3} \end{pmatrix} . \tag{369}$$

The contractions $q \to 1$ is especially transparent here with expansion

$$L = I + 2i\gamma \begin{pmatrix} S^3 & S^- \\ S^+ & -S^3 \end{pmatrix} + \dots \tag{370}$$

for $\gamma \to 0$, where S^3, $S^\pm$ are nondeformed spin variables.

The entries of matrix L satisfy the relations which can be cast in the matrix form

$$L^1 R_-^{-1} L^2 R_- = R_+^{-1} L^2 R_+ L^1 . \tag{371}$$

There exists the generalized trace (so called q-trace)

$$\mathrm{tr}_q A = \mathrm{tr}(DA) \tag{372}$$

with

$$D = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} , \tag{373}$$

such that for any matrix A we have

$$\mathrm{tr}_q^2 R^{-1} A^2 R = \mathrm{tr}_q \mathrm{I}^1 A \ , \tag{374}$$

where we calculate tr_q only over the second factor in the tensor product of two auxiliary spaces.

From these properties we see immediately that

$$[\mathrm{tr}_q L,\ L] = 0 \ , \tag{375}$$

i.e., $\mathrm{tr}_q L$ plays the role of Casimir and we shall denote it as C.

The explicit calculation gives

$$C = \left(q + q^{-1}\right)\left(q^{2S^3} + q^{-2S^3}\right) + \left(q - q^{-1}\right)^2 \left(S^+S^- + S^-S^+\right) \ , \tag{376}$$

or

$$C = q^{2S^3+1} + q^{-2S^3-1} + 2\left(q - q^{-1}\right)^2 S^-S^+ \ , \tag{377}$$

so that in the irreducible representations of spin s the eigenvalues of C are given by $q^{2s+1} + q^{-2s-1}$. This prompts the introduction of the operator J such that

$$C = q^{2J+1} + q^{-2J-1}, \tag{378}$$

which is an analogue of J from the usual $sl(2)$.

Armed with this knowledge we can attack the problem of local hamiltonian in a way analogous to that of section 8.

We shall take two spin operators and describe them by the corresponding L-operators, which we shall denote by $L_\pm$ and $M_\pm$. Let $L(x)$ and $M(y)$ be the corresponding Lax operator, taken in the form (350). The FCR we shall use to find the representation for the $R^{s,s}(x)$ Lax operator $R(x)$ looks as follows

$$L(1/x)\, M(1/y)\, R(x/y) = R(x/y)\, M(1/y)\, L(1/x) . \tag{379}$$

Putting $R(x)$ into the form

$$R(x) = \mathrm{P}r(x) \tag{380}$$

we have instead

$$M(1/x)\,L(1/y)\,r(x/y) = r(x/y)\,M(1/y)\,L(1/x)\,. \tag{381}$$

Altogether (381) comprises four matrix equations. Taking r to be a function of invariant of spin S and T we trivially satisfy two of them. One of remaining looks as follows

$$\left(x^{-1}M_+L_- + xM_-L_+\right)r(x) = r(x)\left(xM_+L_- + x^{-1}M_-L_+\right). \tag{382}$$

We shall take lower left element of this equation

$$\left(x^{-1}q^{-T^3}S^+ + xq^{S^3}T^+\right)r(x) = r(x)\left(xq^{-T^3}S^+ + x^{-1}q^{S^3}T^+\right) \tag{383}$$

and consider it in the subspace

$$q^{T^3}S^+ + T^+q^{-S^3} = 0 \;\;, \tag{384}$$

where the invariant takes the form (378) with

$$J = T^3 + S^3 \;\;. \tag{385}$$

We shall look for $r(x)$ as a function of q^{2J+1}. Our equation looks as follows

$$\left(x^{-1}q^{-J} - xq^J\right)q^{S^3}S^+r(x,q^{2J+1}) = r(x,q^{2J+1})\left(xq^{-J} - x^{-1}q^J\right)q^{S^3}S^+. \tag{386}$$

Using the commutation relation

$$S^+q^{2J+1} = q^{-2}q^{2J+1}S^+ \tag{387}$$

we transform the equation (386) into the functional equation

$$\frac{r(qw)}{r(q^{-1}w)} = \frac{1-x^2w}{x^2-w} \quad , \tag{388}$$

where

$$w = q^{2J+1}. \tag{389}$$

We shall not discuss this equation here; suffice to say, that it is a q-deformation of equation (253) and its solution is given by some q-deformation of Γ-function (known also as q-exponent and exp of quantum dilogarithm). This finishes the general discussion of the XXZ model.

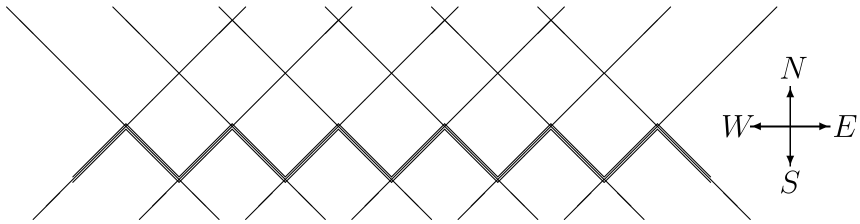

Figure 1: Discrete space time and initial saw

11 Inhomogeneous chains and discrete time shift

Here I shall present some development of the general scheme of Algebraic Bethe Anzatz which will allow to include more dynamical models under its spell. Until now two main observables — momentum and energy were treated differently. We used discretized space, but continuous time; thus we introduced a finite space shift U and infinitesimal generator of time shift H. To make our consideration more manifest invariant it is natural to discretize also time. Exactly this will be done here. We used lattice as a way of regularization and always had in mind corresponding continuous limit. But some people take discrete space-time more seriously as an inevitable consequence of gravity. I shall not open this discussion here sticking to the formal aspects only.

The shift operator U was obtained above as a trace of the monodromy at some distinguished value of spectral parameter. We need now two such distinguished values. The way to achieve this goal is to consider a specially inhomogenous chain with the spectral parameter taking alternating values $\lambda \pm \omega$ for some fixed ω. This simple idea indeed works as we shall see momentarily.

Let $L_{n,f}(\lambda)$ be a fundamental Lax operator with the same quantum space h_n and auxiliary space V_f. The monodromy of the inhomogeneous chain is given by

$$T_f(\lambda, \omega) = L_{2N,f}(\lambda + \omega)L_{2N-1,f}(\lambda - \omega) \ldots L_{2,f}(\lambda + \omega)L_{1,f}(\lambda - \omega). \tag{390}$$

We want to argue, that

$$U_+ = \mathrm{tr}_f T_f(\omega, \omega), \quad U_- = \mathrm{tr}_f T_f(-\omega, \omega) \tag{391}$$

play the role of shifts in the characteristic (light-like) directions in the discrete space-time. This space-time is natural to draw as shown in Figure 1, where the monodromy is taken as a product along the initial saw, depicted by a fat line.

The space runs from east to west and time from south to north. The $L_{2n,f}(\lambda+\omega)$ is transport along NW direction and $L_{2n+1,f}(\lambda - \omega)$ is the same in SW direction.

To see the shift properties of operators $U_\pm$ we consider even more inhomogeneous monodromy inserting into the string of Lax operators $L_{n,f}(\lambda \pm \omega)$ the Lax operator $L^{-1}_{f,a}(\mu - \lambda)$, treating the space a as quantum and f as auxiliary. This admixture is put between $L_{2n,f}(\lambda + \omega)$ and $L_{2n-1,f}(\lambda - \omega)$. The new monodromy looks as follows

$$T_f(\lambda, \omega|a, n, \mu) = L_{2N,f}(\lambda + \omega)L_{2N-1,f}(\lambda - \omega) \ldots \tag{392}$$
$$\ldots L_{2n,f}(\lambda + \omega)L^{-1}_{f,a}(\mu - \lambda)L_{2n-1,f}(\lambda - \omega) \ldots L_{2,f}(\lambda + \omega)L_{1,f}(\lambda - \omega).$$

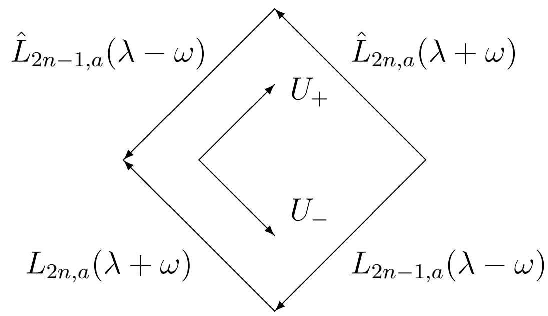

Figure 2: Zero curvature equation

Due to the similar FCR for all entries here this monodromy also satisfies FCR and its trace

$$F(\lambda,\omega|a,n,\mu) = \mathrm{tr}_f T_f(\lambda,\omega|a,n,\mu) \tag{393}$$

gives a commuting family as function of λ.

Now let us look at $F(\lambda,\omega|a,n,\mu)$ at distinguished points $\lambda = \pm\omega$. We use the normalization

$$L_{n,f}(0) = \mathrm{P}_{n,f} \quad , \tag{394}$$

where P is a permutation of h_n and V_f, which coincide as spaces. We have

$$\begin{aligned} F(\omega,\omega|a,n,\lambda) = \mathrm{tr}_f\, &(L_{2N,f}(2\omega)P_{2N-1,f}\ldots \\ &\ldots L_{2n,f}(2\omega)L^{-1}_{f,a}(\lambda-\omega)P_{2n-1,f}\ldots L_{2,f}(2\omega)P_{1,f}\Big) \,. \end{aligned} \tag{395}$$

Note, that we changed parameter μ to λ. Due to relation

$$L^{-1}_{f,a}(\lambda-\omega)\mathrm{P}_{2n-1,f} = \mathrm{P}_{2n-1,f}L^{-1}_{2n-1,a}(\lambda-\omega) \tag{396}$$

we can bring $L^{-1}_{f,a}(\lambda-\omega)$ to extreme right to get

$$F\left(\omega,\omega|a,n,\lambda\right) = U_+ L^{-1}_{2n-1,a}(\lambda-\omega). \tag{397}$$

Analogously

$$F\left(-\omega,\omega|a,n,\lambda\right) = L^{-1}_{2n,a}(\lambda+\omega)U_-. \tag{398}$$

The commutativity of F in the LHS of (397) and (398) leads to the equation

$$U_+L^{-1}_{2n-1,a}(\lambda-\omega)L^{-1}_{2n,a}(\lambda+\omega)U_- = L^{-1}_{2n,a}(\lambda+\omega)U_-U_+L^{-1}_{2n-1,a}(\lambda-\omega)\ , \tag{399}$$

which we can rewrite due to commutativity of U_+ and U_- in the form

$$L_{2n,a}(\lambda+\omega)L_{2n-1,a}(\lambda-\omega) = U_-L_{2n-1,a}(\lambda-\omega)U_-^{-1}\ U_+^{-1}L_{2n,a}(\lambda+\omega)U_+. \tag{400}$$

This equation has natural interpretation as a zero curvature condition for the transport around the elementary plaquette in our space-time. On Figure 2 we use notations

$$\widehat{L}_{2n-1,f}(\lambda-\omega) = U_-L_{2n-1,f}(\lambda-\omega)U_-^{-1} \quad ; \tag{401}$$

$$\widehat{L}_{2n,f}(\lambda+\omega) = U_+^{-1}L_{2n,f}(\lambda+\omega)U_+\ . \tag{402}$$

Thus we see, that U_+ is a shift in NE direction and U_-^{-1} is shift in NW direction. In terms of shifts in N (time) and W (space) directions e^{-iH} and e^{-iP} we have

$$U_+ = e^{-i(H-P)/2} \ , \quad U_- = e^{i(H+P)/2} \ . \tag{403}$$

This is our main assertion. Now I shall describe a more explicit expressions for $U_\pm$. Using our usual Ansatz

$$L_{n,f}(\lambda) = \mathrm{P}_{n,f} l_{n,f}(\lambda) \tag{404}$$

we get

$$U_+ = \mathrm{tr}_f \left(\mathrm{P}_{2N,f} l_{2N,f}(2\omega) \mathrm{P}_{2N-1,f} \ldots \mathrm{P}_{2,f} l_{2,f}(2\omega) \mathrm{P}_{1,f} \right) \tag{405}$$

and bringing all $l_{n,f}(2\omega)$ to right we get

$$U_+ = V \prod_n l_{2n,2n-1}(2\omega) \ \ , \tag{406}$$

where V^{-1} is a shift $n \to n+1$.

Analogously we have

$$U_- = \prod_n l_{2n,2n-1}(-2\omega) V \ \ , \tag{407}$$

$$l_{2n,2n-1}(-\lambda) = l_{2n,2n-1}^{-1}(\lambda) \quad . \tag{408}$$

From identification (403) we get

$$e^{iP} = U_+ U_- = V^2 \quad ; \tag{409}$$

$$\begin{aligned} e^{-iH} \ &= \ U_+ U_-^{-1} = V \prod l_{2n,2n-1}(2\omega) V^{-1} \prod l_{2n,2n-1}(2\omega) = \\ &= \ \prod l_{2n+1,2n}(2\omega) \prod l_{2n,2n-1}(2\omega) \ . \end{aligned} \tag{410}$$

We see, that the physical space shift is shift $n \to n+2$; in other words two lattice sites of our space lattice constitute one physical site. We already have seen such trick in the discussion of nonlinear σ-model.

The derivation of the BAE for the inhomogeneous chain does not differ from that given above. The equations look like

$$\left(\frac{\alpha(\lambda_j + \omega)\alpha(\lambda_j - \omega)}{\delta(\lambda_j + \omega)\delta(\lambda_j - \omega)} \right)^N = \prod_{k \neq j} S(\lambda_j - \lambda_k) \ \ , \tag{411}$$

where $\alpha(\lambda)$ and $\delta(\lambda)$ are local eigenvalues and factor $S(\lambda)$ in the RHS is a quasiparticle phase factor. From this we read the quasiparticle momentum and energy:

$$e^{ip} = \frac{\alpha(\lambda + \omega)\alpha(\lambda - \omega)}{\delta(\lambda + \omega)\delta(\lambda - \omega)}; \quad e^{ih} = \frac{\delta(\lambda + \omega)\alpha(\lambda - \omega)}{\alpha(\lambda + \omega)\delta(\lambda - \omega)} \quad . \tag{412}$$

These expressions nicely turn into our previous formulas in the limit $\omega \to 0$. This finishes our description of a general scheme and I turn to examples.

12 Examples of dynamical models in discrete space-time

I shall present two examples for the general scheme of the previous section. The first is associated with XXX_s spin chain. The BAE

$$\left(\frac{(\lambda_j+\omega+is)(\lambda_j-\omega+is)}{(\lambda_j+\omega-is)(\lambda_j-\omega-is)}\right)^N = \prod_{k\neq j}^{l} \frac{\lambda_j-\lambda_k+i}{\lambda_j-\lambda_k-i} \tag{413}$$

can be investigated as above. The spectrum around the antiferromagnetic state, already described in section 8, leads to the dispersion law for the excitations

$$p(\lambda) = \operatorname{arctg}\sinh\pi(\lambda+\omega) + \operatorname{arctg}\sinh\pi(\lambda-\omega)\ ; \tag{414}$$

$$\epsilon(\lambda) = \frac{1}{\cosh\pi(\lambda+\omega)} + \frac{1}{\cosh\pi(\lambda-\omega)}\ . \tag{415}$$

In the limit $\omega\to\infty$ we get the relativistic one particle spectrum

$$p(\lambda) = m\ \sinh\pi\lambda\ , \quad \epsilon(\lambda) = m\ \cosh\pi\lambda\ , \tag{416}$$

where m is obtained via the "dimensional transmutation"

$$m = \frac{1}{\Delta}e^{-\pi\omega} \tag{417}$$

if we introduce lattice spacing Δ to anticipate the continuous limit.

Let us concentrate on the spin of the excitations. The results of section 8 are rather complicated. However they drastically simplify in the limit $s\to\infty$. Indeed the restriction $a\le 2s$ for kink labels disappears and the sequence $0, a_1, \ldots, a_{2n-1}, 0$ can be considered as parametrizing a particular singlet, entering the representation of $sl(2)$ in $\prod^{2n}\otimes\mathrm{C}^2$. For $n=1$ (two particle state) we have just one such state $(0,1,0)$, corresponding to a singlet in $\mathrm{C}^2\otimes\mathrm{C}^2$. The phase factor, corresponding to the triplet state in spin space and singlet in the kink space is given by the $s\to\infty$ limit of (281) and (279) and is rather simple:

$$S(\lambda) = S_{\mathrm{t}}(\lambda)\frac{\lambda-i}{\lambda+i}S_{\mathrm{t}}(\lambda)\ \ , \tag{418}$$

where $S_{\mathrm{t}}(\lambda)$ is given by (211). The last expression is nothing, but the phase factor for a (triplet) $\otimes$ (singlet) state with respect to two independent $sl(2)$ groups with S-matrix, given by a tensor product of two S-matrices of type $S^{1/2,1/2}$.

Now we mention, that

$$sl(2)\otimes sl(2) = o(4) \tag{419}$$

with $\mathrm{C}^2\otimes\mathrm{C}^2$ being a vector representation for it. In other words, we can interpret the excitations in our inhomogeneous XXX_s model in the limit $s\to\infty$ as vector particles corresponding to the vector representation of $o(4)$.

This particle content coincides exactly with what is believed to be true in the S^3 nonlinear σ-model (or the $sl(2)$ principal chiral model). The inhomogeneous XXX_s spin model just realizes one particular sector of $sl(2)$ chiral model in the limit

$s \to \infty$; only particles in singlet state with respect to the right spin appear. However this is enough to characterize the full S-matrix.

There are more reasons to justify this correspondence, in particular the exact calculation of the β-function of the renormalization group is possible via the BAE. I do not have time to discuss it here.

The second example is the Sine-Gordon model. This dynamical system was instrumental for the developing the ABA by Sklyanin, Takhtajan and me in the end of 70-ties. From that time there were developed quite a few approaches to investigate it via BAE. Here I present an approach based on alternating inhomogeneous chain, developed by A. Volkov and me at 1992.

As the dynamical variables on an alternating lattice I shall use the Weyl variables u_n, v_n with the exchange relations

$$u_n v_n = q v_n u_n, \quad q = e^{i\gamma} \ . \tag{420}$$

The auxiliary Lax operator

$$L_{n,a}(x) = \begin{pmatrix} u_n & x v_n \\ -x v_n^{-1} & u_n^{-1} \end{pmatrix} \tag{421}$$

belongs to the XXZ family with FCR (344). I cannot help mentioning, that in simplicity formula (421) beats even the expression for the Lax operator of XXX model (32).

The local product of two Lax operators $L_{2n,a}(x\kappa)L_{2n-1,a}(x\kappa^{-1})$ corresponds to transport along a physical lattice site. We have explicitly the matrix

$$\begin{pmatrix} u_{2n}u_{2n-1} - x^2 v_{2n} v_{2n-1}^{-1} & x\left(\kappa^{-1}u_{2n}v_{2n-1} + \kappa v_{2n}u_{2n-1}^{-1}\right) \\ -x\left(\kappa v_{2n}^{-1}u_{2n-1} + \kappa^{-1}u_{2n}^{-1}v_{2n-1}^{-1}\right) & u_{2n}^{-1}u_{2n-1}^{-1} - x^2 v_{2n}^{-1}v_{2n-1} \end{pmatrix} . \tag{422}$$

We can identify this matrix with the XXZ Lax operator of type (342) with the spin variables q^{S^3}, $S^{\pm}$ given by (22), (23) if we impose the constraint

$$\tilde{w}_n = u_{2n}u_{2n-1}v_{2n}v_{2n-1}^{-1} = 1 \tag{423}$$

at each physical lattice site. This constraint reduces the number of local degrees of freedom from two to one, as it must be.

Indeed, let us put

$$e^{i\pi_n} = u_{2n}v_{2n-1} \ ; \tag{424}$$
$$e^{i\varphi_n} = u_{2n}u_{2n-1} \ . \tag{425}$$

Then (422) turns to (342) after division by x and similarity transform with matrix

$$D = \begin{pmatrix} \kappa^{1/2} & 0 \\ 0 & \kappa^{-1/2} \end{pmatrix} , \tag{426}$$

if we put

$$m^2 = \kappa^2 \ . \tag{427}$$

After this identification it is natural to assume that the BAE look like (411). The derivation is nontrivial, because the representation for the q-deformed spin variables is not of the highest weight. However it can be done and the local factors $\alpha(\lambda)$ and $\delta(\lambda)$ correspond to spin $-1/2$. Returning to the additive variables $x = e^{\lambda}$, $\kappa = e^{\omega}$ we have the BAE in the form

$$\left(\frac{\sinh\left(\lambda_j+\omega+\frac{i\gamma}{2}\right)\sinh\left(\lambda_j-\omega+\frac{i\gamma}{2}\right)}{\sinh\left(\lambda_j+\omega-\frac{i\gamma}{2}\right)\sinh\left(\lambda_j-\omega-\frac{i\gamma}{2}\right)}\right)^{N/2} = \prod_{\substack{k=1\\k\neq j}}^{l} \frac{\sinh(\lambda_j-\lambda_k-i\gamma)}{\sinh(\lambda_j-\lambda_k+i\gamma)} \, . \tag{428}$$

Now I turn to the description of the time shift operator. For that we are to find the fundamental Lax $L_{n,f}(x)$ operator, corresponding to the auxiliary $L_{n,a}(x)$ from (421). We write the equation

$$L_2\,(1/x)\,L_1\,(1/y)\,r\,(x/y) = r\,(x/y)\,L_1\,(1/y)\,L_2\,(1/x) \tag{429}$$

with the natural brief notations

$$L_1(x) = L_{n_1,a}(x), \quad L_2(x) = L_{n_2,a}(x) \, . \tag{430}$$

The solution will be looked for as a function of the variable

$$w = u_2 u_1 v_2^{-1} v_1 \, . \tag{431}$$

The diagonal matrix elements of equations (429) are trivially satisfied. The right upper matrix element of the equation (429) looks as follows

$$(x u_2 v_1 + v_2 u_1^{-1}) r(x,w) = r(x,w)(u_2 v_1 + x v_2 u_1^{-1}) \tag{432}$$

and can be reduced to the functional equation

$$\frac{r(x,qw)}{r(x,q^{-1}w)} = \frac{xw+1}{x+w} \tag{433}$$

by using the exchange relations (420). We have already seen this equation in section 10. Indeed, for the representation (22), (23) we can check, that w coincides with q^{2J+1}.

Denoting

$$w_{2n} \;=\; u_{2n} u_{2n-1} v_{2n}^{-1} v_{2n-1} \quad ; \tag{434}$$

$$w_{2n+1} \;=\; u_{2n+1} u_{2n} v_{2n+1}^{-1} v_{2n} \, , \tag{435}$$

we have the expression for the time evolution operator

$$e^{-iH} = \prod r(\kappa^2, w_{\text{even}}) \prod r(\kappa^2, w_{\text{odd}}) \, . \tag{436}$$

This expression is simple enough to allow us to write down the equation of motion explicitly. It is convenient to introduce a new set of variables ψ_n, such that w_n are their "multiplicative derivatives"

$$w_n = \frac{\psi_{n+1}}{\psi_{n-1}} \, . \tag{437}$$

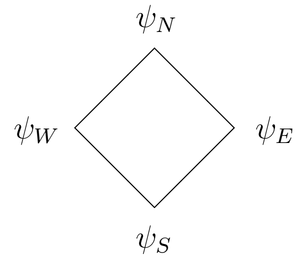

Figure 3: Elementary plaquette

Variables ψ_m and ψ_n commute for $n-m$ even, so there is no problem in order of factors in (437). The exchange relations among ψ_n are nonlocal, but it is not of concern to us; it suffices to say, that given ψ_n does not commute with only one w, namely w_n, and for this pair there is the Weyl relation

$$\psi_n w_n = q^2 w_n \psi_n \ . \tag{438}$$

We can take the set $\{\psi_{\text{even}}, w_{\text{even}}\}$ or $\{\psi_{\text{odd}}, w_{\text{odd}}\}$ as a set of independent Weyl variables in the physical Hilbert space. I skip here the subtle question of boundary conditions for integrating (437), which is settled in the original literature.

Equations of motion are especially simple in ψ variables. Let $\hat{\psi}_n$ be a variable ψ_n once displaced in time

$$\hat{\psi}_n = e^{iH} \psi_n e^{-iH} \quad . \tag{439}$$

Take for definiteness ψ_{odd}, i.e. ψ_{2n+1}. Only one factor, containing w_{2n+1} in (436), does not commute with it, so we have using the functional equation (433)

$$\begin{aligned} \hat{\psi}_{2n+1} &= r^{-1}(w_{2n+1})\, \psi_{2n+1}\, r(w_{2n+1}) = \psi_{2n+1} r^{-1}(q^{-2} w_{2n+1}) r(w_{2n+1}) = \\ &= \psi_{2n+1} \frac{\kappa^2 q^{-1} w_{2n+1} + 1}{\kappa^2 + q^{-1} w_{2n+1}} = \psi_{2n+1} \frac{\kappa^2 q^{-1} \psi_{2n+2} + \psi_{2n}}{q^{-1} \psi_{2n+2} + \kappa^2 \psi_{2n}} \ . \end{aligned} \tag{440}$$

This is our equation of motion. For ψ_{even} the derivation is exactly the same. On the discrete space–time drawn on Figure 1 the equation (440) connects the ψ attached to an elementary plaquette shown in Figure 3 and can be rewritten as

$$\psi_N = \psi_S \frac{\kappa^2 q^{-1} \psi_W + \psi_E}{q^{-1} \psi_W + \kappa^2 \psi_E} \tag{441}$$

or

$$(q^{-1} \psi_N \psi_W - \psi_S \psi_E) = \kappa^2 (q^{-1} \psi_S \psi_W - \psi_N \psi_E) \ . \tag{442}$$

The last simple quadratic equation looks quite appealing.

Let us show, how the Sine–Gordon equation appears in the formal continuous limit. We also confine ourselves to classical commuting variables.

The naive prescription that ψ_n define a continuous function of space–time variables does not work. We have already seen a similar situation in the case of NLS model. The way out is to use variables

$$\chi = \left\{ \begin{array}{c} \psi \\ \psi^{-1} \end{array} \right. \tag{443}$$

alternatively on each second SE characteristic line of our lattice. Then equations of motion (441) transform into

$$\chi_N = \chi_S^{-1} \frac{\kappa^2 q^{-1} \chi_W \chi_E + 1}{q^{-1} \chi_W \chi_E + \kappa^2} \tag{444}$$

and for large κ^2 and classical limit $q = 1$ we have

$$\frac{\chi_N \chi_S}{\chi_E \chi_W} = 1 + \frac{1}{\kappa^2} \left(\frac{1}{\chi_E \chi_W} - \chi_E \chi_W \right) + \dots . \tag{445}$$

We put here

$$\chi = e^{i\varphi} \tag{446}$$

and consider φ as defining a smooth function of x, t in the continuous limit. Then evidently

$$\frac{\chi_N \chi_S}{\chi_E \chi_W} = \exp \left\{ i \frac{\Delta^2}{2} (\varphi_{tt} - \varphi_{xx}) + \dots \right\} \tag{447}$$

and introducing the rescaling

$$\frac{1}{\kappa^2} = m^2 \Delta^2 \tag{448}$$

we get from (445) the Sine–Gordon equation

$$\varphi_{tt} - \varphi_{xx} + 2m^2 \sin 2\varphi = 0 . \tag{449}$$

In quantum version the scaling (448) is to be modified, accounting to the mass renormalization. The investigation of BAE (428) allows to describe this renormalization exactly, adding to the nonperturbative intuition in Quantum Field Theory. However I cannot speak about it here and stop technical developments to come to some conclusions.

13 Conclusions and perspectives

The first and the last formulas in this text being the same, our exposition closed the circle. We have learned the technique of Algebraic Bethe Ansatz for solving integrable models and have shown, how it works in detail on the simplest example of spin 1/2 XXX magnetic chain. Several other models were treated more superficially, only the specific details were given. Several parameters, appearing in these generalizations: spin s, anisotropy parameter γ, shift ω in the alternating chain, allow to include in our treatment most known examples of soliton theory, including relativistic model of Quantum Field Theory. Thus the spin chains showed their great universality. If we add, that the fashionable models of conformal field theory also can be included as particular limits, then we could claim, that we deal with the classifying object in the theory of integrable models.

We treated here only rank 1 case associated with the Lie algebra $sl(2)$. Generalization to the higher rank simple Lie algebras is possible and many results are already known. In particular, one can use a Cartan subalgebra label to introduce one extra dimension for space.

We did not make a stress on the subject of quantum groups or quantum symmetries, as this was discussed in other lectures at the School. However I suppose, that I made it clear, how quantum groups were created naturally inside of ABA (of course, their appearance in the C^* algebra approach of Professor Woronowitz was completely independent).

Discrete space-time approach to quantum integrable models seems to me most promising and elegant. It has already allowed to write explicitely the quantum equation of motion. I cannot help feeling that it is just the beginning of analysis, including complex analysis, on discrete manifolds.

From pure mathematical point of view many of reasonings in my lectures were only heuristic. The statements were "claims" rather than "propositions" not to mention "theorems". I am not a supporter of absolute rigor in mathematical physics, but would be happy, if some of my claims become better justified. Even more important is the fact, that new mathematical constructions and results could appear in course of this justification. Especially promising are problems of infinite tensor products in antiferromagnetic case and their connections with infinite–dimensional Lie algebras and analysis on the discrete space-time. I leave these challenges to my listeners.

14 Comments on the literature on BAE

Here I will add some comments which will serve as a guide to the literature on Bethe Ansatz. Only seminal (from my own point of view) papers will be mentioned. The recent monograph [1] contains quite vast list of references. Another monograph [2] reflects the developments around Bethe Ansatz mostly prior to the advent of ABA.

BAE were first written by H. Bethe in [3] for the spin 1/2 XXX model. He did not use the uniformizing rapidity variables, which apparently appeared first in a paper by Takahashi [4]. First integral equation for the distribution of roots was derived by L. Hulthen [5]. My exposition of the spin 1/2 XXX model follows closely my paper with Takhtajan [6]. The statement, that the spin of a spin wave is 1/2 was announced by us in [7]. The scattering theory for the excitations over the ferromagnetic vacuum was developed by Babbit and Thomas [8].

General formalism of the ABA was worked out by Sklyanin, Takhtajan and me in [9] on the example of the Sine–Gordon model. The state of art at the end of 70–ties was described in my survey [10].

The q–deformed $sl(2)$ algebra with defining relations (12), (13) was introduced by Kulish and Reshetikhin in 1981 in [11] in connection with the higher spin XXZ model. More general investigation of the quadratic algebras was performed by Sklyanin [12], who pointed out the connection with the Hopf algebras [13]. The proper algebraic understanding of general structures, appearing here, was formulated by Jimbo [14] and in the most perfect form by Drinfeld [15]. The role of the fundamental Lax operator was underlined by Tarasov, Takhtajan and me in [16]. There we followed the ideas of fusion, introduced by Kulish, Reshetikhin and Sklyanin in [17], where the formula (254) was derived.

Investigation of BAE for higher spin XXX model was done by Takhtajan [18], (see also Babujan [19]). The kink interpretation of the excitations was developed by Reshetikhin in [20]. The thorough investigation of BAE for NLS model with chemical potential was done by Lieb and Liniger [21].

The interpretation of the NLS and SG models as XXX and XXZ chains, correspondingly, was done by Izergin and Korepin [22]. Korepin was also the first to propose the formula of type (211) for the matrix element of S-matrix for the excitations above Dirac sea [23].

The functional equation for the fundamental Lax operator (388) for XXZ model has a natural place in the theory of $\widehat{sl_q}(2)$ algebra, see e.g. [24].

The idea of imbedding the S^2 σ-model into XXX chain belongs to Haldane [25] and Takhtajan and me [26]. More detailed development was done by Affleck, who in particularly stressed the role of the topological ϑ-term [27]. The BAE (338) was derived by Bytsko (unpublished).

The idea of the usefulness alternating inhomogenous chain to treat the relativistic models was underlined by Reshetikhin and me [28] in our treatment of the principal $sl(2)$ chiral model. It was developed further by Destri and De-Vega [29]. The zero curvature interpretation was given by Volkov and me in [30].

The quantum equations of motion (442) for the SG model appeared in [31]; they coincide in the classical limit ($q = 1$) with Hirota equations [32]. In terms of variables f_A, connected with ψ_A by

$$f_A - f_B = \frac{1}{\psi_A \psi_B} \tag{450}$$

in the form

$$\frac{(f_N - f_E)(f_W - f_S)}{(f_N - f_W)(f_E - f_S)} = \kappa^4 \quad , \tag{451}$$

as was commented by Volkov [33]. This makes the appearance of discrete complex analysis indispensable. The same equations were recently advertised by Capel et al. [34]. Another line of thought on discrete geometry and discrete SG equation belongs to Pinkal et al. [35]. Quadratic algebras appear now in many instances and guises. I mention the discretized affine algebra, introduced by Semenov-Tjan-Shansky and discussed in some detail in [36].

I finish by reference to my previous lecture course [37] where many aspects of these lectures were treated, and to lecture notes of my Schladming lectures on new applications of BAE to Hofstadter model from condensed matter physics, to high energy QCD and to Liouville model of Conformal Fiel Theory.

In my lectures I did not refer at all to the parallel development in classical statistic physics. The general equivalence exists between 1+1 dimensional quantum dynamical systems and 2 dimensional models of classical statistical physics. Integrable models of the former domain correspond to the exactly soluble models in the latter. A lot of well known results here are connected with the names of Onsager, C. N. Yang, Lieb, Baxter and others. I can refer to the monograph of Baxter [38], where this theme is displaced in great detail.

Another connection with statistical physics is the use of the integrable hamiltonians to define the corresponding Hibbs state $Z^{-1}e^{-\beta H+\mu N}$. This was pioneered in paper of C. P. Yang and C. N. Yang [39], which led to important development, called the Thermodynamic Bethe Ansatz. Already mentioned paper [19] of Babujan was an important step in formulating this technical development.

Returning to quantum field theory interpretation I must say, that until now ABA has given in the way to investigate mass spectrum and S-matrix. More detailed off shell characteristics of systems under consideration are very scarce. The main success was achieved by Smirnov in the discussion of formfactors of local operators [40].

Another line of thought, originated by Korepin, is partly described in [1]. This topic is the most important from the point of view of mathematical physics and physical applications.

References

[1] V.Korepin, N.Bogoliubov, A.Izergin "Quantum Inverse Scattering Method and Correlation Functions", Cambridge monographs on Mathematical Physics, Cambridge University Press, 1993.

[2] M.Gaudin, La function d'Onde de Bethe, Masson, Paris 1983.

[3] H.Bethe Z. Physik, **71** (1931) 205.

[4] M.Takahashi, Progress Theor. Phys., **46** (1971) 401.

[5] L.Hulthen. Arkiv for Matematik Astronomi och Fysik, 26A (1938), 1-106.

[6] L.Takhtajan, L.Faddeev, Zapiski Nauchnych Seminarov LOMI, **109**, (1981) 134, English translation J. Sov. Math. **24** (1984) 241

[7] L.Faddeev, L.Takhtajan, Phys. Lett. A, **85** (1981) 375.

[8] D.Babbitt, L.Thomas, J. Math. Phys.,**19** (1978) 1699.

[9] E.Sklyanin, L.Takhtajan, L.Faddeev, Theor. Math. Phys. **40** (1979) 194

[10] L.Faddeev, Sov. Sci. Reviews, Harwood Academic, London, C1 (1980) 107.

[11] P.Kulish, N.Reshetikhin, Zapiski Nauch. Semin. LOMI, **101** (1981) 101.

[12] E.Sklyanin, Func. Anal. Appl., **16** (1983) 263; **17** (1983) 273.

[13] E.Sklyanin, Uspekhy Math. Sci. **40** (1985) 214.

[14] M. Jimbo, LMPh, **10** (1985) 63.

[15] V.Drinfeld "Quantum groups", in Proceedings of ICM, Berkeley, AMS(1987) p.798.

[16] V.Tarasov, L.Takhtajan, L.Faddeev, TMPh, **57** (1983) 1059.

[17] P.Kulish, N.Reshetikhin, E.Sklyanin, Lett. Math. Phys. **5** (1981) 393.

[18] L.Takhtajan, Phys. Lett. **87** A (1982) 479.

[19] H.Babujan, Nucl. Phys. **B215** (1983) 317.

[20] N.Reshetikhin, J.Phys. A Math. Gen. 24 (1991) 3299.

[21] E.Lieb, W.Liniger, Phys. Rev. **130** (1963) 1605; 1616.

[22] A.Izergin, V.Korepin, LMPh, **6** (1984) 241.

[23] V. Korepin, TMPh, **41** (1979) 953.

[24] I.Frenkel, N.Reshetikhin CMP, **146** (1992) 1.

[25] F.Haldane, Phys. Rev. Lett. 50, (1983) 1153.

[26] L.Faddeev, L.Takhtajan, Preprint E-4-83, Leningrad Steklov Mathematical Institute, (1983).

[27] I.Affleck, J. Phys. Condens. Matter 1 (1989) 3047.

[28] L.Faddeev, N.Reshetikhin, Ann. Phys. **167** (1986) 227.

[29] C.Destri, H. De Vega, Nucl. Phys., B**290** (1987) 363.

[30] A.Volkov, L.Faddeev, TMPh, **92** (1992) 207.

[31] L.Faddeev, A.Volkov, LMPh, **32** (1994) 125.

[32] R.Hirota, Proc. Phys. Soc., Japan, **43** (1977) 2079.

[33] A.Volkov, preprint University of Uppsala; hep-th/9509024.

[34] W.Capel in "100 years of KdV equation", Elsevier, Amsterdam, 1995.

[35] A.Bobenko, N.Kunz, U.Pinkal, SFB-282 preprint, TU-Berlin, 1993.

[36] L.Faddeev In "New Symmetry Principles in Quantum Field Theory", et J.Fröhlich et al, NATO ASI Series, Physics vol.295, (1992) p.159.

[37] L.Faddeev, Int. J. Mod. Phys. **10** (1995) 1845.

[38] R.Baxter, "Exactly Solved Models in Statistical Mechanics", Academic Press, London–New-York, 1982.

[39] C.N.Yang, C.P.Yang, J.Math.Phys. **10** (1969) 1115.

[40] F.Smirnov, Formfactors in Completely Integrable Models in Quantum Field Theory, World Scientific, Singapore, 1992.