

High energy QCD as a completely integrable model

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Abstract

We show that the one-dimensional lattice model proposed by Lipatov to describe the high energy scattering of hadrons in multicolor QCD is completely integrable. We identify this model as the XXX Heisenberg chain of noncompact spin $s = 0$ and find the conservation laws of the model. A generalized Bethe ansatz is developed for the diagonalization of the hamiltonian and for the calculation of hadron-hadron scattering amplitude.

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1. Introduction

Recently it was suggested by Lipatov [1] that the asymptotic behavior of the hadron-hadron scattering amplitudes in QCD in the limit of large invariant energy s and fixed transferred momentum t can be described by means of the quantum inverse scattering method [2, 3]. In the generalized leading logarithmic approximation [4, 5, 6] the scattering amplitude is given by the contributions of the Feynman diagrams having a conserved number n of reggeized gluons in the t -channel [7, 8]. For fixed n the scattering amplitude can be decomposed into t -channel partial waves f_ω , describing n gluon to n gluon scattering with total angular momentum $j = 1 + \omega$. The partial wave satisfies the Bethe-Salpeter equation whose solutions can be expressed in terms of the wave functions χ of the compound states of n gluons as follows [4, 9]:

$$f_\omega(\{\vec{b}_i\}, \{\vec{b}'_i\}) = \sum_\alpha \int d^2b_0 \chi_\alpha(\{\vec{b}_i\}; \vec{b}_0) \chi_\alpha^*(\{\vec{b}'_i\}; \vec{b}_0)$$

Here, $\chi_\alpha(\{\vec{b}_i\}; \vec{b}_0)$ is the wave function of the compound state of n reggeized gluons with transverse coordinates $\{\vec{b}_i\} = \vec{b}_1, \dots, \vec{b}_n$ and it is parameterized by quantum numbers α and by two-dimensional vector \vec{b}_0 which represents the center of mass coordinate of the states. In the special case $n = 2$, the function $\chi(\vec{b}_1, \vec{b}_2; \vec{b}_0)$ is the wave function of the BFKL pomeron [10, 12]. Let us define for all impact vectors $\vec{b}_i = (x_i, y_i)$, $i = 1, \dots, n$, the following complex coordinates: $z_i = x_i + iy_i$ and $\bar{z}_i = x_i - iy_i$ and similarly for the vectors \vec{b}'_i .

Then, in the large N limit, for an arbitrary number n of gluons, the wave functions φ are the eigenstates of the holomorphic and antiholomorphic hamiltonians [1, 11, 9]

$$H_n \chi(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0) = \varepsilon_n \chi(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0), \quad \bar{H}_n \chi(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0) = \bar{\varepsilon}_n \chi(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0), \quad (1)$$

with eigenvalues related to complex angular momentum via $\omega = -\frac{\alpha_s}{4\pi} N(\varepsilon_n + \bar{\varepsilon}_n)$. The hamiltonians H_n and \bar{H}_n describe nearest neighbors interaction of n particles

$$H_n = \sum_{k=1}^n H_{k,k+1}, \quad \bar{H}_n = \sum_{k=1}^n \bar{H}_{k,k+1} \quad (2)$$

with periodic boundary conditions $H_{n,n+1} = H_{n,1}$, $\bar{H}_{n,n+1} = \bar{H}_{n,1}$ and the two-particle hamiltonians are giving by the equivalent representations

$$\begin{aligned} H_{j,k} &= P_j^{-1} \log(z_j - z_k) P_j + P_k^{-1} \log(z_j - z_k) P_k + \log(P_j P_k) + 2\gamma_E \\ &= 2 \log(z_j - z_k) + (z_j - z_k) \log(P_j P_k) (z_j - z_k)^{-1} + 2\gamma_E \end{aligned} \quad (3)$$

where $P_j = i \frac{\partial}{\partial z_j}$ and γ_E is the Euler constant. The same operator can also be represented as [11]

$$H_{i,k} = \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1) - \bar{L}_{ik}^2} - \frac{2}{l+1}, \quad \bar{L}_{ik}^2 = -(z_i - z_k)^2 \partial_i \partial_k \quad (4)$$

Thus, in the large N limit the original problem of the calculation of partial waves is reduced to the diagonalization of two hamiltonians corresponding to spin chain models defined on a one-dimensional lattice with the number of sites equal to the number of reggeized gluons in the t -channel. However, there are additional constraints on the eigenstates of the QCD hamiltonians imposed by the condition that the partial wave f_ω is invariant under holomorphic and antiholomorphic conformal transformations of the coordinates z and \bar{z} [12]. In terms of the wave functions χ which depend only on parts of the coordinates, this condition leads to the following transformation properties

$$\chi(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0) \rightarrow \chi(\{z'_i, \bar{z}'_i\}; z'_0, \bar{z}'_0) = (cz_0 + d)^{2h} (\bar{c}\bar{z}_0 + \bar{d})^{2\bar{h}} \chi(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0) \quad (5)$$

under conformal transformations $z \rightarrow z' = \frac{az+b}{cz+d}$ and $\bar{z} \rightarrow \bar{z}' = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}$ with $ad - bc = \bar{a}\bar{d} - \bar{b}\bar{c} = 1$. The possible values of the conformal weights h and \bar{h} can be parameterized as follows [12, 4, 9]

$$h = \frac{1+m}{2} - i\nu, \quad \bar{h} = \frac{1-m}{2} - i\nu \quad (6)$$

for arbitrary real ν and integer m .

It was suggested [1] that these models can be exactly solved by means of the Bethe ansatz. Among other things, this would imply that the model contains a family of “hidden” integrals of motion. Indeed, at least one nontrivial integral of motion A was found [1]. It commutes with the hamiltonian and is given by

$$A = z_{12}z_{23} \cdots z_{n1}P_1P_2 \cdots P_n, \quad [H_n, A] = 0 \quad (7)$$

together with the analogous operator for the antiholomorphic system. It was noticed [1], that this operator belongs to the family of n mutually commuting operators generated by a monodromy matrix which satisfies the Yang-Baxter equation. However it was not clear that all these operators commute with the hamiltonian and that they are the integrals of motion of the model. In the present paper we show that this is indeed the case. We prove that the model (1) is completely integrable and discuss the possibility of solving it by means of the Bethe ansatz.

2. High energy multicolor QCD as XXX Heisenberg magnet

Let us show that the hamiltonians (3) and (4) coincide with that of the XXX Heisenberg magnet for spin $s = 0$. To give meaning to the limit $s = 0$ we recall that the quantum inverse scattering method allows one [13] to construct a family of exactly solvable XXX models for arbitrary complex values of the spin s . For arbitrary complex s the spin operators S_k^α ($\alpha = 1, 2, 3$) in all sites $k = 1, \dots, n$ of the lattice can be realized as follows

$$S_k^+ = z_k^2 \partial_k - 2sz_k, \quad S_k^- = -\partial_k, \quad S_k^3 = z_k \partial_k - s \quad (8)$$

The definition of the model is based on the existence of a fundamental matrix $R_{f_1 f_2}(\lambda)$ which obeys the Yang-Baxter equation and acts in the auxiliary space $V_{f_1} \otimes V_{f_2} = \mathfrak{h} \otimes \mathfrak{h}$ with the space \mathfrak{h} having the dimension of the local quantum space in each site. For arbitrary complex spin s it is given by the following expression [14]

$$R_{f_1 f_2}(\lambda) = f(s, \lambda) \frac{\Gamma(i\lambda - 2s)\Gamma(i\lambda + 2s + 1)}{\Gamma(i\lambda - J)\Gamma(i\lambda + J + 1)} \quad (9)$$

where the operator J is defined in the space $V_{f_1} \otimes V_{f_2}$ as a solution of the operator equation

$$J(J + 1) = 2\vec{S} \otimes \vec{S} + 2s(s + 1)$$

Notice that the definition of the R -matrix contains ambiguity in multiplication by an arbitrary c -number valued function $f(s, \lambda)$ which is fixed by imposing additional normalization conditions on $R_{f_1 f_2}(\lambda)$. For the special value $\lambda = 0$ of the spectral parameter the operator $R_{f_1 f_2}(0)$ is proportional to the permutation operator $P_{f_1 f_2}$ on $V_{f_1} \otimes V_{f_2}$ and both operators are equal to each other for finite dimensional representations of the spin operators [14] provided that $f(s, 0) = 1$. The hamiltonian of the XXX model of spin s is given by the general expression (2) with the two-particle hamiltonian defined as

$$H_{12} = \frac{1}{i} \frac{d}{d\lambda} \log R_{f_1 f_2}(\lambda) \Big|_{\lambda=0} \quad (10)$$

Let us consider the expression for the hamiltonian H in the limit $s = 0$. Using the expressions (9) and (10) we find that one of the Γ -functions in the numerator of (9) diverges as $\lambda, s \rightarrow 0$. However we use the possibility to choose $f(s, \lambda) = \Gamma(i\lambda + 2s + 1)/\Gamma(i\lambda - s)$ to avoid this problem and obtain the two-particle hamiltonian as

$$H_{ik} = -\psi(-J_{ik}) - \psi(J_{ik} + 1) + 2\psi(1) \quad (11)$$

Here $\psi(x) = d \log \Gamma(x) / dx$ and the operator J_{ik} is one of the solutions of the equation

$$J_{ik}(J_{ik} + 1) = 2\vec{S}_i \otimes \vec{S}_k = -(z_i - z_k)^2 \partial_i \partial_k$$

where we substituted the explicit form (8) of the spin operators for $s = 0$. Comparing the last relation with (4) we notice that the operator $J_{ik}(J_{ik} + 1)$ coincides with the definition of the operator L_{jk}^2 . Substituting

$L_{jk}^2 = J_{jk}(J_{jk} + 1)$ into (4) one can perform the summation over l to get an expression for the hamiltonian which is identical to that in (11)! Thus, we identify the holomorphic and antiholomorphic QCD hamiltonians (1) as hamiltonians of a one-dimensional XXX Heisenberg model with spin $s = 0$. In what follows, we study only the holomorphic sector of the model. The generalization to the antiholomorphic sector is straightforward.

The identification of the model as XXX magnet means that the system (1), which describes high-energy asymptotics in multi-color QCD, is exactly solvable. To find the family of local integrals of motion of the model we follow the standard procedure. To any site k we assign auxiliary and fundamental Lax operators [14]

$$L_{k,a}(\lambda) = \lambda I_n \otimes I_a + i\vec{S}_k \otimes \vec{\sigma}_a, \quad L_{k,f}(\lambda) = R_{k,f}(\lambda) \quad (12)$$

where $\vec{\sigma}$ are the Pauli matrices. The operators $L_{k,a}$ and $L_{k,f}$ act locally in the space $\mathfrak{h}_k \otimes \mathbb{C}^2$ and $\mathfrak{h}_k \otimes \mathfrak{h}$, respectively, where \mathfrak{h}_k is the quantum space in the k -th site and the dimensions of \mathfrak{h} and \mathfrak{h}_k coincide. Taking the ordered product of the Lax operators along the lattice we define the auxiliary monodromy matrix as a matrix in \mathbb{C}^2

$$T_a(\lambda) = L_{n,a}(\lambda)L_{n-1,a}(\lambda)\dots L_{1,a}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad (13)$$

and analogously the fundamental monodromy matrix $T_f(\lambda)$. Taking the trace of the monodromy matrices over the auxiliary space we get two operators, the auxiliary and fundamental transfer matrices,

$$\Lambda(\lambda) = \text{tr}_a T_a(\lambda), \quad \tau(\lambda) = \text{tr}_f T_f(\lambda) \quad (14)$$

which act in the full quantum space of the model and commute with each other for different values of the spectral parameters [2]

$$[\tau(\lambda), \Lambda(\mu)] = [\tau(\lambda), \tau(\mu)] = [\Lambda(\lambda), \Lambda(\mu)] = 0 \quad (15)$$

Differentiating the both sides of this relation with respect to the spectral parameters λ and μ and putting $\lambda = \mu = 0$ we will get a family of mutually commuting conservation laws of the model. Moreover, the fundamental transfer matrix $\tau(\lambda)$ contains the local integrals of motion including the hamiltonian of the model, while the operator $\Lambda(\lambda)$ allows one to construct their eigenstates by means of the Bethe ansatz. The explicit form of the local integrals of motion is given by [14]

$$I_k = \frac{1}{i} \frac{d^k}{d\lambda^k} \log \tau(\lambda) \Big|_{\lambda=0} = \frac{1}{i} \frac{d^k}{d\lambda^k} \log \text{tr}_f T_f(\lambda) \Big|_{\lambda=0}, \quad k = 1, 2, \dots \quad (16)$$

where the k -th operator describes the interaction between $k+1$ nearest neighbors on the lattice. In particular, the operator $I_1 = H = \sum_{k=1}^n H_{k,k+1}$ coincides with the hamiltonian of the model.

To find the explicit form of the operator $\Lambda(\lambda)$ we substitute the spin operators (8) for $s = 0$ into the standard form of the auxiliary Lax operator (12) and represent the result as a matrix in \mathbb{C}^2

$$L_{k,a}^{(s=0)}(\lambda) = \lambda I + i v_k \otimes \bar{u}_k \partial_k \quad (17)$$

where $\partial_k \equiv \frac{\partial}{\partial z_k}$ and $v_k = \begin{pmatrix} 1 \\ z_k \end{pmatrix}$ and $\bar{u}_k = (z_k, -1)$ are vectors in the auxiliary space \mathbb{C}^2 . After substitution of (17) into (13) we use the relation $\bar{u}_k v_j = (z_k - z_j)$ to get

$$\Lambda_{s=0}(\lambda) = 2\lambda^n - S(S+1)\lambda^{n-2} + Q_3\lambda^{n-3} + \dots + Q_n \quad (18)$$

where the operator S is defined as

$$S(S+1) = - \sum_{k>j} z_{kj}^2 \partial_k \partial_j = \vec{S} \vec{S} = S^3 S^3 + \frac{1}{2}(S^+ S^- + S^- S^+)$$

and \vec{S} is the total spin of lattice. The operators Q_k for $k = 3, \dots, n$ are given by

$$Q_k = \sum_{n \geq i_1 > i_2 > \dots > i_k \geq 1} i^k z_{i_1 i_2} z_{i_2 i_3} \dots z_{i_k i_1} \partial_{i_1} \partial_{i_2} \dots \partial_{i_k}$$

and we recognize the operator Q_n as the operator A defined in (7). Notice, that in the expansion of $\Lambda_{s=0}(\lambda)$ the term with λ^{n-1} is absent due to the orthogonality of vectors v_k and \bar{u}_k . It follows immediately from (15) that the integrals of motions I_k and Q_j are mutually commuting operators

$$[I_k, Q_j] = [Q_k, Q_j] = [I_k, I_j] = 0$$

with $I_1 = H$ being the holomorphic hamiltonian.

To diagonalize these operators one may try to apply the algebraic Bethe ansatz [2]. However, the attempt to use the Bethe ansatz for $s = 0$ fails from the very beginning. The algebraic Bethe ansatz is based on the existence of the highest weights in each site of the lattice. Trying to define them, we find that for $s = 0$ the equations $S_k^+|\omega_k\rangle = 0$ and $S_k^3|\omega\rangle = s|\omega\rangle$ have a trivial solution $|\omega_n\rangle = const$ which is annihilated by the spin operator \vec{S} . The reason why it happens is the following. As we will show in sect.4 the spin operators form for $s = 0$ the representation $t^{0,2}(g)$ of the principal series of the noncompact group $SL(2, \mathbb{C})$. This is a unitary infinite dimensional representation [15] which does not contain a highest weight, in contrast with unitary finite dimensional representations of $SU(2)$. Thus, to diagonalize the hamiltonian of the model one has to generalize the algebraic Bethe ansatz to the case of noncompact groups. Nevertheless, there is a trick which allows one to apply the algebraic Bethe ansatz to find special eigenstates of the model.

3. Algebraic Bethe ansatz for spin $s = -1$

There is one-to-one correspondence between the XXX models of spin $s = 0$ and $s = -1$ based on the following relation between the Lax operators (12) in both models

$$L_{k,a}^{(s=-1)}(\lambda) = \lambda I + i\partial_k v_k \otimes \bar{u}_k = \left(L_{k,a}^{(s=0)}(\lambda) \right)^T = P_k L_{k,a}^{(s=0)}(\lambda) P_k^{-1}$$

where operator T denotes similarity transformation. The fundamental Lax operators (12) satisfy the same relation. From this property and the definition (13) we find the relation between the monodromy matrices of the models

$$T_a^{(s=-1)}(\lambda) = \left(T_a^{(s=0)}(\lambda) \right)^T = P_1 P_2 \dots P_n T_a^{(s=0)}(\lambda) (P_1 P_2 \dots P_n)^{-1}$$

and analogously for $T_f^{(s=-1)}(\lambda)$. It is obvious from (14) that the same relations are valid between the transfer matrices but in this case one can use the fact that they commute with the operator $Q_n \equiv A$ defined in (7) to write

$$\Lambda_{s=-1}(\lambda) = \Lambda_{s=0}^T(\lambda) = (z_{12}z_{23} \dots z_{n1})^{-1} \Lambda_{s=0}(\lambda) z_{12}z_{23} \dots z_{n1} \quad (19)$$

and analogously for $\tau_{s=-1}(\lambda)$. Thus, the transfer matrices in the XXX models of spin $s = 0$ and $s = -1$ have the same eigenvalues and their eigenstates are related as follows

$$|\varphi(z_i; z_0)\rangle = z_{12}z_{23} \dots z_{n1} |\hat{\varphi}(z_i; z_0)\rangle \quad (20)$$

As follows from (10) the same relation (19) holds between the hamiltonians of both models.

The reason why we included the XXX model of spin $s = -1$ into consideration is that one is able to apply the algebraic Bethe ansatz for spin $s = -1$.¹ In contrast with spin $s = 0$, it is possible to find a nontrivial highest weight for spin $s = -1$. Indeed, by substituting the explicit form of the spin operator (8) for $s = -1$ into the equations $S_k^+|\omega_k\rangle = 0$ and $S_k^3|\omega_k\rangle = -|\omega_k\rangle$ we find that it has the nontrivial solution $|\omega_k\rangle = 1/z_k^2$ which allows one to construct the pseudovacuum state as

$$|\Omega\rangle = \frac{1}{z_1^2 z_2^2 \dots z_n^2}$$

To define the Bethe states one has to use the B operator for spin $s = -1$ defined in (13). Then the Bethe states are given by

$$|\hat{\varphi}_l(\{\lambda\})\rangle = B(\lambda_1)B(\lambda_2) \dots B(\lambda_l) \frac{1}{z_1^2 z_2^2 \dots z_n^2} \quad (21)$$

¹However, as is shown in sect.4, the algebraic Bethe ansatz gives us only special set of the eigenstates

where the parameters $\{\lambda\} = (\lambda_1, \dots, \lambda_l)$ are solutions of the Bethe equation for $s = -1$

$$\left(\frac{\lambda_k - i}{\lambda_k + i}\right)^n = \prod_{j=1, j \neq k}^n \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i} \quad (22)$$

and the operator B is defined through (13) as an element of the auxiliary monodromy matrix corresponding to spin $s = -1$. The Bethe states (21) depend on a positive integer number l which is fixed by the additional constraint (5). Indeed, using (20) it can be easily shown that the transformation properties (5) of an eigenstate of the XXX hamiltonian for $s = 0$ are equivalent to the following conditions for the eigenstates of the XXX hamiltonian for spin $s = -1$

$$S^3|\hat{\varphi}(z_i; 0)\rangle = -h|\hat{\varphi}(z_i; 0)\rangle, \quad S^+|\hat{\varphi}(z_i; 0)\rangle = 0, \quad |\hat{\varphi}(z_i; z_0)\rangle = |\hat{\varphi}(z_i - z_0; 0)\rangle \quad (23)$$

where \vec{S} is the total spin of the lattice for $s = -1$. Comparing these constraints with the properties of the Bethe states, $S^3|\hat{\varphi}\rangle = -(n+l)|\hat{\varphi}\rangle$ and $S^+|\hat{\varphi}\rangle = 0$, we find that

$$n + l = h, \quad l = 0, 1, \dots$$

where h was defined in (6) and n is the number of sites, or equivalently, the number of reggeized gluons in the t -channel. This relation means that the algebraic Bethe ansatz allows us to construct a very special class of eigenstates corresponding to *positive integer values* of the conformal weight h of the n -gluon wave function. To find the remaining states for arbitrary complex h a generalization of the Bethe ansatz is required.

Using the relations (21) and (20) we finally get the expression for the Bethe states which diagonalize the original hamiltonian as follows

$$|\varphi_l(z_i; 0; \{\lambda\})\rangle = z_{12}z_{23} \dots z_{n1} B(\lambda_1)B(\lambda_2) \dots B(\lambda_l) \frac{1}{z_1^2 z_2^2 \dots z_n^2}$$

The integrals of motion for the XXX model with spin $s = -1$ are related to those for XXX model with spin $s = 0$ by equations identical to equation (19) between the transfer matrices. That is why they have the same eigenvalues in both models. The explicit expressions for the eigenvalues of integrals of motions for arbitrary spin s have been found in algebraic Bethe ansatz [14] and we use these expressions for $s = -1$ to get

$$I_k = \sum_{j=1}^l \frac{1}{i} \frac{d^k}{d\lambda_j^k} \log \frac{\lambda_j + i}{\lambda_j - i} \quad (24)$$

Note that this expression was found with the function $f(s, \lambda)$ that defines the normalization of the R -matrix in (9) chosen to be equal to one. It follows from (16), that for arbitrary $f(s, \lambda)$ the expression (24) gets an additional trivial contribution $-in \frac{d^k}{d\lambda^k} f(s, \lambda)|_{\lambda=0}$. Given a solution to the Bethe equation (22) this relation yields the spectrum of local integrals of motion in the original model with hamiltonian H . In particular, the eigenvalues of the hamiltonian are equal to

$$\varepsilon_n \equiv I_1(\lambda) = -2 \sum_{k=1}^l \frac{1}{\lambda_k^2 + 1}$$

where $\{\lambda\}$ obey the Bethe equation (22) for a fixed number of reggeized gluons n and $s = -1$. Thus, for special values of the conformal weights h the algebraic Bethe ansatz constructed in this subsection yields the wave function of the model (1) and the corresponding eigenvalue of the QCD hamiltonian.

4. Eigenstates as highest weights of representations of $SL(2, \mathbb{C})$

So far we have considered only the properties of the holomorphic hamiltonian in (1). If we know its eigenstates we can generalize them for the antiholomorphic hamiltonian in (1) and construct the solution of the system (1) as the product of holomorphic and antiholomorphic eigenstates

$$\chi(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0) = \varphi(z_i; z_0) \bar{\varphi}(\bar{z}_i; \bar{z}_0)$$

Let us combine the holomorphic and antiholomorphic hamiltonian to define the following operator acting on both z and \bar{z} coordinates of particles:

$$\mathcal{H} = H_n^T + \bar{H}_n$$

where similarity transformation was defined in (19). This hamiltonian has the following properties. As we have seen in sect.3, the hamiltonian H_n describes the XXX magnet with the generalized spin $s = 0$ while the hamiltonian H_n^T corresponds to the XXX magnet with spin $s = -1$. Recall that the hamiltonians H_n and H_n^T are related to each other by an equation similar to (19) and they have the same spectrum of eigenvalues and the corresponding eigenstates satisfy relation (20). Hence, for fixed ε_n and $\bar{\varepsilon}_n$ in (1) the hamiltonian \mathcal{H} has the eigenvalue $\varepsilon_n + \bar{\varepsilon}_n$ and the corresponding eigenstate is

$$\hat{\chi}(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0) = \hat{\varphi}(z_i; z_0) \bar{\varphi}(\bar{z}_i; \bar{z}_0) = (z_{12} z_{23} \dots z_{n1})^{-1} \chi(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0) \quad (25)$$

Thus, there is one-to-one correspondence between this function and the eigenstate χ . We notice that the hamiltonian \mathcal{H} is selfadjoint and its eigestates $\hat{\chi}$ are orthogonal to each other. The reason why we consider the functions $\hat{\chi}(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0)$ is that these functions are the highest weight of the irreducible infinite dimensional unitary representation of the principal series of the $SL(2, \mathbb{C})$ group.

To show this we recall that the unitary representation of the principal series, $t^{\rho, m}(g)$, is realized on the space of square integrable functions $f(z, \bar{z})$ according to

$$t^{\rho, m}(g)f(z, \bar{z}) = (cz + d)^{-k/2 + i\rho/2 - 1} (\bar{c}\bar{z} + \bar{d})^{k/2 + i\rho/2 - 1} f\left(\frac{az + b}{cz + d}, \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}\right)$$

where k is an integer, ρ is a real number and the scalar product in this space is given by

$$\langle f_1 | f_2 \rangle = \int dz d\bar{z} f_1(z, \bar{z}) f_2^*(z, \bar{z})$$

In this representation the group generators can be realized as holomorphic spin operators (8) with $s = -1/2 - k/4 + i\rho/4$ and antiholomorphic spin operators with $\bar{s} = -1/2 + k/4 + i\rho/2$.

For the one-dimensional model with the hamiltonian \mathcal{H} we have in each site k of the lattice holomorphic spin operators S_k^\pm and S_k^3 defined by (8) for $s = -1$ and antiholomorphic spin operators \bar{S}_k^\pm and \bar{S}_k^3 given by expressions (8) for $s = 0$ with the replacement of z by \bar{z} . These six operators commute with the hamiltonian \mathcal{H} (and not with the QCD hamiltonian $H_n + \bar{H}_n$!) and form the representation $t^{0,2}$ of the principal series of the $SL(2, \mathbb{C})$ group which we identify as a local quantum space in the k -th site. The full spin operators of the lattice $S^\alpha = \sum_{k=1}^n S_k^\alpha$ and $\bar{S}^\alpha = \sum_{k=1}^n \bar{S}_k^\alpha$ act on the tensor product of these representations which is reducible with respect to $SL(2, \mathbb{C})$ and which can be decomposed into the direct sum of the irreducible unitary representations of the principal series as follows. It is well known [15] that the product of two infinite dimensional unitary representations of the principal series, $t^{\rho_1, 2m_1}$ and $t^{\rho_2, 2m_2}$, where m_1 and m_2 are integer and ρ_1 and ρ_2 are real, is decomposed into the direct sum of the irreducible representations of the same principal series $t^{\rho_3, 2m_3}$ with m_3 being integer. Applying this result to the tensor product of n copies of $t^{0,2}$ corresponding to all site of the lattice we get

$$\underbrace{t^{0,2} \otimes t^{0,2} \otimes \dots \otimes t^{0,2}}_n = \bigoplus_{\rho, m} t^{\rho, 2m}$$

that is the total quantum space of the lattice is decomposed into the sum of infinite dimensional unitary representations of the principal series of $SL(2, \mathbb{C})$ with real ρ and integer m . In principle, one may try to evaluate the multiplicities and the Clebsch-Gordan coefficients for each $t^{\rho, 2m}$ in this decomposition using the standard formulas [15]. Thus, the eigenstates $\hat{\chi}(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0)$ of the hamiltonian \mathcal{H} belong to the representations of the principal series $t^{\rho, 2m}$. However, there are additional constraints on the functions $\hat{\chi}(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0)$ imposed by the transformation properties (5), or equivalently by the conditions (23) for holomorphic states and similar relations for antiholomorphic states. Using the relation (25) we represent these conditions in the following form

$$S^3|\hat{\chi}\rangle = -h|\hat{\chi}\rangle, \quad S^+|\hat{\chi}\rangle = 0, \quad \bar{S}^3|\hat{\chi}\rangle = -\bar{h}|\hat{\chi}\rangle, \quad \bar{S}^+|\hat{\chi}\rangle = 0$$

where $\hat{\chi} \equiv \hat{\chi}(\{z_i, \bar{z}_i\}; z_0 = 0, \bar{z}_0 = 0)$ and the conformal weights h and \bar{h} were defined in (6). Using the explicit form of h and \bar{h} we conclude that the eigenstates of the hamiltonian \mathcal{H} which obey these additional conditions are the highest weights of the irreducible representation $t^{4\nu, 2m}$ of the principal series of the $SL(2, \mathbb{C})$. Thus, for arbitrary complex conformal weights h and \bar{h} in (6) there might exist the eigenstate of the model. On the other hand, as was shown in sect.3, the algebraic Bethe ansatz is applicable only for integer h and it cannot be generalized even for noninteger conformal weights.

5. Generalized Bethe ansatz

To find the eigenstates corresponding to arbitrary complex values of the conformal weights one could use the method of the Q -operator [16] proposed by Baxter. Using the relations between the XXX models of spins $s = 0$ and $s = -1$ we restrict ourselves only by consideration of the model with $s = -1$. In this case, there exists an operator $Q(\lambda)$ which acts on the full quantum space of the lattice and obeys the Baxter equation

$$\Lambda(\lambda)Q(\lambda) = (\lambda - i)^n Q(\lambda - i) + (\lambda + i)^n Q(\lambda + i), \quad [Q(\lambda), \Lambda(\mu)] = [Q(\lambda), Q(\mu)] = 0 \quad (26)$$

Thus, the operators Λ and Q can be diagonalized simultaneously. Their corresponding eigenvalues are c -number valued functions of the spectral parameter λ which obey the same equation (26). That is why we are using below the same notations $\Lambda(\lambda)$ and $Q(\lambda)$ for the eigenvalues of these operators. Solving this functional equation and using the analyticity properties of $\Lambda(\lambda)$ as a function of λ , one is able to find the eigenvalues of the auxiliary transfer matrix $\Lambda(\lambda)$ and the operator $Q(\lambda)$. The standard way of solving the Baxter equation consists of choosing eigenvalues $Q(\lambda)$ to depend polynomially on the spectral parameter λ

$$Q(\lambda) = \text{const.} \prod_{k=1}^l (\lambda - \lambda_k) \quad (27)$$

where $\{\lambda\} = \{\lambda_1, \dots, \lambda_l\}$ are arbitrary complex numbers and l is positive integer. The parameters $\{\lambda\}$ are found by imposing analyticity conditions for the function $\Lambda(\lambda)$ at $\lambda = \lambda_k$. Namely, substituting $\lambda = \lambda_k$ into the both sides of the Baxter equation we get an equation for the parameters $\{\lambda\}$ which coincides with the Bethe equation (22). Thus, the algebraic Bethe ansatz corresponds to the special case of the eigenvalues of the Q -operator as finite polynomials. To go beyond the algebraic Bethe ansatz one has to look for solutions of the Baxter equation among the functions more general than polynomials. The simplest possibility which is realized in the Toda model is to search for the solution among the entire functions having *infinite* number of zeros, $l \rightarrow \infty$.

Using the definition (18) we find that the function $\Lambda(\lambda)$ is a polynomial of power n in λ with the coefficients given by the operators $S(S+1)$ and Q_k . This suggests the following way to solve the Baxter equation. First, one substitutes the general form of the function $\Lambda(\lambda)$ into the l.h.s. of the Baxter equation, replacing all operators by their unknown eigenvalues, S and Q_k ($k = 3, \dots, n$). Second, one solves the functional equation (26) and finds the solution for $Q(\lambda)$ which depends on these quantum numbers. Third, imposing the condition that $Q(\lambda)$ is an entire function of λ one determines the possible values of S and Q_k and substitutes them into (18) to find the eigenvalue of the auxiliary transfer matrix.

Thus, for any fixed allowed set of the quantum numbers $\{S, Q_k\}$ we will find the corresponding function $Q(\lambda)$ satisfying the Baxter equation. It is important to notice that among these functions there are very special ones (27) corresponding to the Bethe ansatz solution of the XXX model of spin $s = -1$. Moreover, in this particular case we know the explicit form of the eigenstates (21) and eigenvalues (24) of the local integrals of motion. All these expressions, including the Q -function (27), explicitly depend on the parameters $\{\lambda_k\}$ which are solutions of the Bethe equation (22). The natural question appears: is it possible to express both the eigenstates (21) and the eigenvalues (24) in terms of $Q(\lambda)$ in such a way, that all their dependence on $\{\lambda_k\}$ will be contained inside the Q -function? In this case, one will get a unique possibility to perform an “analytical continuation” of the results found within the framework of the algebraic Bethe ansatz by replacing the special values (27) of the Q -function by a general solution of the Baxter equation.

Let us express the Bethe states (21) in terms of the Q -function defined in (27). Substituting the explicit form of the auxiliary Lax operator into the definitions (13) one finds that for $s = -1$ the operator $B(\lambda)$ is a polynomial of power $n - 1$ in λ which can be represented in the following form

$$B(\lambda) = iS^-(\lambda - x_1)(\lambda - x_2) \dots (\lambda - x_{n-1}) \quad (28)$$

where x_1, \dots, x_{n-1} are operator zeros of $B(\lambda)$. The precise meaning of x_k as roots of the operator polynomial $B(\lambda)$ was given in the framework of the functional Bethe ansatz [17, 18]. These operators have the following properties

$$[S^-, x_k] = [S^3, x_k] = [x_k, x_j] = 0$$

which allows us not to worry about their ordering in (28). Now we may rewrite the Bethe states (21) in terms

of the operators x_k

$$|\hat{\phi}_l\rangle = \prod_{j=1}^l iS^- \prod_{k=1}^{n-1} (\lambda_j - x_k)|\Omega\rangle = (iS^-)^l \prod_{k=1}^{n-1} \prod_{j=1}^l (\lambda_j - x_k)|\Omega\rangle$$

where λ_j satisfies the Bethe equation (22) for $s = -1$ and fixed l and n . Now we immediately recognize that the product over j coincides with the expression (27) for the function $Q(\lambda)$ defined for $\lambda = x_k$. Notice that the Q -operator doesn't commute with the operators x_k and the notation $Q(x_k)$ means the eigenvalue of the Q -operator defined to be evaluated for operator value of the spectral parameter. Thus, the Bethe states can be expressed in terms of the eigenvalues of the Q -operator as follows

$$|\hat{\phi}\rangle = (iS^-)^{h-n} Q(x_1)Q(x_2)\dots Q(x_{n-1}) \frac{1}{z_1^2 z_2^2 \dots z_n^2} \quad (29)$$

where h is the conformal weight defined in (6). The remarkable property of this expression is that it does not depend explicitly on the parameters λ_k . The operators S^- and x_k are determined by the properties of the model and are the same for all eigenstates. Different eigenstates $|\hat{\phi}_l\rangle$ are parameterized by different solutions $Q(\lambda)$ of the Baxter equation. After being written in this form, the relations (29) admit a natural analytic continuation to arbitrary complex values of S , or equivalently h . Moreover, these states satisfy the additional conditions (23) which follow from the analytical continuation of the analogous relations for the Bethe states.

Once we know the eigenvalues of the Q -operator the eigenvalues of the auxiliary transfer matrix can be easily found from (26). To find the eigenvalues of the local integrals of motion we use their expressions found by the algebraic Bethe ansatz method, rewrite them in terms of the Q -functions and perform analytical continuation. Using equation (24) we represent the eigenvalues of the operator I_k for $s = -1$ in the following form

$$I_k = \frac{1}{i} \sum_{j=1}^l \frac{d^k}{d\lambda_j^k} \log \frac{\lambda_j + i}{\lambda_j - i} = \frac{1}{i} \frac{d^k}{d\lambda^k} \log \prod_{j=1}^l \frac{\lambda_j + i + \lambda}{\lambda_j - i + \lambda} \Big|_{\lambda=0}$$

and in terms of Q -function (27) this expression looks like

$$I_k = \frac{1}{i} \frac{d^k}{d\lambda^k} \log \frac{Q(-\lambda - i)}{Q(-\lambda + i)} \Big|_{\lambda=0} \quad (30)$$

Although this relation was found as a relation between eigenvalues of the operators I_k and Q it can be extended to be an operator relation because both operators have the same eigenstates.

6. Solution of the Baxter equation

To find the eigenstates and eigenvalues of the integrals of motion one has to solve the Baxter equation for $s = -1$ and substitute the solution for the function $Q(\lambda)$ into the relations (29) and (30). Using the general form of the function $\Lambda(\lambda)$ in terms of quantum numbers $\{S, Q_k\}$ we get the following functional equation for the function $Q(\lambda)$

$$(2\lambda^n - h(h-1)\lambda^{n-2} + Q_3\lambda^{n-3} + \dots + Q_n)Q(\lambda) = (\lambda + i)^n Q(\lambda + i) + (\lambda - i)^n Q(\lambda - i) \quad (31)$$

where we used the relations (23) and (18) to identify the eigenvalue of the operator S with the conformal weight h of the eigenstates defined in (6). In this equation h and Q_k are independent parameters whose values are restricted by analyticity properties of the solution $Q(\lambda)$.

The Baxter equation (31) depends on the set of quantum numbers $\{h, Q_k\}$ and obeys the following property. It is invariant under the replacement

$$h \rightarrow 1 - h \quad (32)$$

Hence, if the function $Q(\lambda; h, \{Q_k\})$ is a solution of the Baxter equation then so is $Q(\lambda; 1 - h, \{Q_k\})$. Combined with (29), this property allows us to relate the eigenstates of the model with the conformal weights h and $1 - h$. Thus, trying to solve the Baxter equation (31) we use the symmetry (32) to restrict the possible values of the conformal weights to the fundamental region

$$\text{Re } h \geq \frac{1}{2} \quad (33)$$

or in terms of the parameters m defined in (6), $m \geq 0$. It is interesting to note that the Bethe ansatz solution (27) for $Q(\lambda)$ corresponds to integer positive h and the symmetry (32) allows us to generalize the results of the sect.3 to arbitrary integer h by putting $l = |h - \frac{1}{2}| - n + \frac{1}{2}$. This is the simplest example of the analytical continuation of results obtained by means of the Bethe ansatz.

In the limit of large λ the Baxter equation can be rewritten as

$$-h(h-1)\lambda^{n-2}Q(\lambda) = (\lambda+i)^nQ(\lambda+i) + (\lambda-i)^nQ(\lambda-i) - 2\lambda^nQ(\lambda) \sim i^2 \frac{d^2}{d\lambda^2}(\lambda^n Q(\lambda))$$

where we neglected nonleading terms in both sides of the equation. Solving the resulting second order differential equation we get the general large- λ asymptotic behavior of the solutions of the Baxter equation in the fundamental region (33)

$$Q(\lambda) \stackrel{\lambda \rightarrow \infty}{\sim} \lambda^{h-n} \equiv \lambda^l \quad (34)$$

Using the asymptotics (34) we recognize the special role of integer positive values of the conformal weights h . The analyticity of $Q(\lambda)$ as a function of λ together with (34) implies that for $h-n \in \mathbb{Z}_+$ the solution of the Baxter equation is polynomial of order $h-n$ in λ which is in accordance with (27).

To solve the Baxter equation (31) for arbitrary λ we perform the Mellin transformation of the function $Q(\lambda)$

$$Q(\lambda) = \int_0^\infty d\omega \omega^{i\lambda-1} Q(\omega) \quad (35)$$

After substitution of (35) into (31), the original finite differential equation for $Q(\lambda)$ is replaced by an ordinary n -th order differential equation for $Q(\omega)$:

$$\left[\frac{(\omega-1)^2}{\omega} \left(i\omega \frac{d}{d\omega} \right)^n - \sum_{k=0}^{n-2} Q_{n-k} \left(i\omega \frac{d}{d\omega} \right)^k \right] Q(\omega) = 0$$

where $Q_2 \equiv -h(h-1)$. A simple analysis shows that the solution of this equation has singularities at $\omega = 1$. Indeed, in the large λ limit the function (35) gets its leading contribution from the integration at the vicinity of the point $\omega = 1$ and the asymptotics (34) imply that

$$Q(\omega) \stackrel{\omega \rightarrow 1}{\sim} (1-\omega)^{-1-h+n}$$

where h takes values in the fundamental region (33). Thus, for the expression (35) to be well defined one has to fix the prescription for the integration of the $\omega = 1$ singularity. Since the integration in (35) is performed along the real positive axis in the complex ω -plane one can deform the integration path at the point $\omega = 1$ to encircle the singularity either in lower or upper half plane. Moreover, rotating both integration paths as $\omega \rightarrow e^{i\pi}\omega$ and $\omega \rightarrow e^{-i\pi}\omega$, respectively, we avoid the singular point $\omega = 1$ and get the following expression for the Q -function

$$Q(\lambda) = (C_+ e^{\pi\lambda} + C_- e^{-\pi\lambda}) \int_0^\infty d\omega \omega^{i\lambda-1} Q(-\omega)$$

where C_+ and C_- are arbitrary constants. However the value of the constants is fixed by the condition that for arbitrary complex λ the integral should be convergent at $\omega = 0$ and $\omega = \infty$

$$C_+ = -C_- = \frac{1}{2} \quad (36)$$

where the numerical value can be arbitrary. Changing the integration variable $z = 1/(1-\omega)$ we obtain

$$Q(\lambda) = \sinh(\pi\lambda) \int_0^1 dz (1-z)^{i\lambda-1} z^{-i\lambda-1} Q(z) \quad (37)$$

where the function $Q(z)$ obeys the following equation

$$\left[\left(-iz(1-z) \frac{d}{dz} \right)^n + z(1-z) \sum_{k=0}^{n-2} Q_{n-k} \left(-iz(1-z) \frac{d}{dz} \right)^k \right] Q(z) = 0$$

Let us consider the solutions of this equation in the special case $n = 2$ corresponding to the compound state of two reggeized gluons - the BFKL pomeron.

For $n = 2$ the equation for the function $Q(z)$ has the following form

$$z(1-z)Q''(z) + (1-2z)Q'(z) + h(h-1)Q(z) = 0$$

where prime denotes differentiation with respect to z . The equation is invariant under the replacement $z \rightarrow 1-z$ and its general solution is well known [19] to be a linear combination of the hypergeometric functions

$$Q(z) = c_1 F(h, 1-h; 1; z) - c_2 F(h, 1-h; 1; 1-z)$$

where c_1 and c_2 are arbitrary constants. After substitution of this expression into (37) we find the Q -function as

$$Q(\lambda) = c_1 Q_0(\lambda) + c_2 Q_0(-\lambda) \quad (38)$$

where $Q_0(\lambda)$ denotes the function

$$Q_0(\lambda) = \sinh(\pi\lambda) \int_0^1 dz (1-z)^{i\lambda-1} z^{-i\lambda-1} F(h, 1-h; 1; z)$$

We use the integral representation for the hypergeometric function [19] in order to express the solution in the following form

$$Q_0(\lambda) = \sinh(\pi\lambda) \sin(\pi h) \int_0^1 dz (1-z)^{i\lambda-1} z^{-i\lambda-1} \int_0^1 dt t^{h-1} (1-t)^{-h} (1-tz)^{h-1} \quad (39)$$

This expression for $Q_0(\lambda)$ can be represented as a double Pochhammer contour integral

$$Q_0(\lambda) = \frac{i}{4} \oint_P dz z^{-i\lambda-1} (z-1)^{i\lambda-1} \oint_P dt t^{h-1} (t-1)^{-h} (1-tz)^{h-1}$$

where the contour P incloses the singular points 0 and 1 in complex z and t planes. Taking (39) and expanding the last factor in the integrand in powers of zt we perform the integration over z and get the following factor $\Gamma(i\lambda)\Gamma(1-i\lambda) = -\frac{i\pi}{\sinh(\pi\lambda)}$ which generates the singularities of the function $Q(\lambda)$ at $\lambda = i\mathbb{Z}$. They originate from the “dangerous” points $z = 0$ and $z = 1$ in the integral. To preserve the analyticity of the Q -function we use the possibility to choose arbitrary values of the constants C_+ and C_- to put them equal to (36). This leads to the appearance of the factor $\sinh(\pi\lambda)$ in (39) which compensates the singularities of the integral. Finally, we get the representation for Q_0 as an infinite sum

$$Q_0(\lambda) = \sum_{k=1}^{\infty} (-)^k \frac{k}{(k!)^3} \frac{\Gamma(h+k) \Gamma(k-i\lambda)}{\Gamma(h-k) \Gamma(1-i\lambda)} = h(h-1) {}_3F_2(1+h, 2-h, 1-i\lambda; 2, 2; z)|_{z \rightarrow 1} \quad (40)$$

where ${}_3F_2$ is the generalized hypergeometric function [19]. This expression has the following properties.

In the k -th term of the sum the λ -dependence comes from the ratio $\Gamma(k-i\lambda)/\Gamma(1-i\lambda)$ which is a polynomial of order k in λ . Hence, being expanded in powers of λ the function $Q_0(\lambda)$ turns out to be an infinite series. This seems to be in contradiction with the fact, that in the special case of positive integer values of $l = h - 2$ the Bethe ansatz gives us an expression for the Q -function which is a polynomial of power $h - 2$ in λ . However, notice that in the representation (40) the factor $1/\Gamma(h-k)$ ensures the truncation of the sum after the $k = (h-1)$ -th term for $h \geq 2$. Moreover, the expression (40) explicitly obey the symmetry (32) which implies that the series terminates also for negative integer h .

For arbitrary noninteger h the expression (40) is an infinite series in λ . Using the properties [19] of the function ${}_3F_2$ one finds that this series converges only if $\text{Im } \lambda < 0$. Then, the general form of the solution (38) implies that the solution of the Baxter equation $Q(\lambda)$ is given by the function $Q_0(\lambda)$ in the lower half plane in λ and by $Q_0(-\lambda)$ in the upper half plane. It is interesting to note that for integer h the function satisfies the relation $Q_0(-\lambda) = (-1)^h Q_0(\lambda)$ which implies that both terms in (38) are equivalent.

After substitution of (40) into (38) we get an expression for $Q(\lambda)$ which is an entire function of λ having an infinite number of zeros. Only in the case of positive integer h the number of zeros is finite. It can be checked that the roots $\{\lambda_k\}$ do obey the Bethe equation for $n = 2$. This is in accordance with general conditions on the Q -function discussed in sect.5.

7. Conclusions

In this paper we found that the one-dimensional lattice model proposed by Lipatov to describe high-energy scattering of hadrons in QCD is completely integrable. Applying the quantum inverse scattering method we identified the Lipatov model as the generalized one-dimensional XXX chain of spin $s = 0$. We found the family of local integrals of motion and developed the generalized Bethe ansatz for their diagonalization. The corresponding eigenstates and eigenvalues, (29) and (30), are expressed in terms of the Q -function which satisfies the Baxter equation (26) for spin $s = -1$. Solving this equation in the special case of the lattice with $n = 2$ sites we found the expressions (38) and (40) for the Q -function. It turns out that after the substitution of this function into (29) and (30) we will get the expressions for the wave function χ of the compound state of $n = 2$ reggeized gluons and the corresponding eigenvalue of the QCD hamiltonian which are identical to that [12] for the BFKL pomeron. The details of the calculations and the proof of the statements made in this paper will be published elsewhere.

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