

Discrete Heisenberg-Weyl Group and Modular Group

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Abstract

It is shown that the generators of two discrete Heisenberg-Weyl groups with irrational rotation numbers θ and $-1/\theta$ generate the whole algebra \mathcal{B} of bounded operators on $L_2(\mathbf{R})$. The natural action of the modular group in \mathcal{B} is implied. Applications to dynamical algebras appearing in lattice regularization and some duality principles are discussed.

Writing a contribution to a memorial volume one always feels the mixture of the admiration for passed away and sorrow of a loss. Because of a difference in age and geographical (and/or political) obstructions I did not have any significant personal scientific encounters with J.Schwinger. But our brief exchanges during several short meetings and most of all reading his papers influenced my way of thinking and writing to a great extent.

Together with his colossal work on QED and general quantum field theory, J.Schwinger did pay attention to technical problem, pertaining to the ordinary nonrelativistic quantum mechanics [1]. So I hope, that the comments in this paper, which are confined to the objects in a simplest Hilbert space of quantum theory, namely $L_2(\mathbf{R})$, still would amuse him. My excuse is that these comments were generated by a dynamical problem in quantum field theory.

What I want to discuss is to my belief the base of several "duality principles", which appear nowadays, i.e., in connection with string theory and conformal field theory. However I shall not go into technicalities of these subjects. More on this will be mentioned in the conclusion of this paper.

I consider a Hilbert space \mathcal{H} , where the usual coordinate and momentum operators Q and P , satisfying the Heisenberg commutation rule

$$[P, Q] = -i\hbar I$$

are irreducibly represented. For example, we can take $\mathcal{H} = L_2(\mathbf{R})$ with elements $\psi(x)$ and usual action of Q and P

$$P\psi(x) = \frac{\hbar}{i} \frac{d}{dx} \psi(x); \quad Q\psi(x) = x\psi(x)$$

(coordinate representation).

Together with Q and P consider their exponentials

$$u = e^{iPq/\hbar}, \quad v = e^{iQp/\hbar}$$

with the Weyl commutation relations

$$uv = e^{2\pi i\theta} vu,$$

where

$$\theta = \frac{pq}{2\pi\hbar}.$$

Here p and q are fixed real numbers with dimensions of momentum and coordinate, respectively, so that θ is dimensionless. I could of course fix the units in such a way, that the dimensions disappear. Only the parameter θ is essential in what follows.

It is clear that (for fixed p and q) the map

$$\{P, Q\} \rightarrow \{u, v\}$$

is not invertible. In other words, whereas P and Q generate the full algebra \mathcal{B} of operators in \mathcal{H} , i.e. via Weyl formula

$$A = \int f(s, t) \exp\left(i \frac{Qs + Pt}{\hbar}\right) \frac{dsdt}{2\pi\hbar},$$

it is not true for u and v . In other words, algebra \mathcal{A} , algebraically generated by u and v , is a proper subalgebra in \mathcal{B} . Indeed, it is evident, that operators

$$\hat{u} = e^{\frac{2\pi i Q}{q}}, \quad \hat{v} = e^{\frac{2\pi i P}{p}}$$

(and so all algebra, generated by them) commute with u and v . The term "algebraically generated" means that \mathcal{A} is obtained by a suitable closure of polynomials in u, v, u^{-1} and v^{-1} . In this sense, while \hat{u} and \hat{v} as elements of \mathcal{B} are functions of u and v

$$\hat{u} = v^{1/\theta}, \quad \hat{v} = u^{1/\theta},$$

they do not belong to \mathcal{A} for generic θ .

A natural question appears, what one is to add to (u, v) to be able to generate all \mathcal{B} . I want to argue, that in case when θ is irrational, it is (u, v, \hat{u}, \hat{v}) which generate all \mathcal{B} .

I came to the question of extension the algebra \mathcal{A} , generated by u and v , in course of work in collaboration with A.Volkov on $U(1)$ lattice current algebra [2]. This algebra is generated by dynamical variables $w_n, n = 1, \dots, N$, with commutation relations

$$w_n w_{n+1} = \omega w_{n+1} w_n$$

$$[w_n, w_m] = 0 \quad |n - m| \geq 2$$

and periodic boundary condition for finite chain. A natural question about the finding of the operators U , realizing the shift

$$w_n \rightarrow w_{n+1},$$

so that

$$w_{n+1} = U w_n U^{-1}$$

is easily reduced to the local problem in the Weyl algebra for a pair of operators u, v with relation

$$uv = \omega vu;$$

one is to construct two operators f and g such that

$$[f, u] = 0; \quad [g, v] = 0,$$

$$vfg = fgv.$$

If we confine our search to algebra \mathcal{A} , generated by u and v , then we are to take

$$g = r(v), \quad f = r(u),$$

where $r(u)$ is an algebraic function of u (we take into consideration the invariance with respect to the shift), and the required property is achieved, if $r(u)$ is a solution of the functional equation

$$\frac{r(\omega u)}{r(u)} = \frac{1}{u}.$$

This equation is easily solved by the series

$$r(u) = \sum \omega^{\frac{n^2-n}{2}} u^n$$

which makes sense either if $|\omega| < 1$ or ω is a root of unity, when we can truncate the series. For arbitrary $\omega = e^{2\pi i\theta}$ with values on the circle, which is needed in applications, this answer is unsatisfactory. The way out of this difficulty is proposed in my Varenna lecture notes [3] and corresponds exactly to the extension of the algebra, where to look for the operators f and g . It was shown, that the operators f and g could be found in algebra generated by u, v, \hat{u}, \hat{v} . More on this is below.

Now let us prove our main assertion.

Lemma *Let θ be irrational. Then u, v, \hat{u}, \hat{v} constitute an irreducible set of generators in $\mathcal{B}(\mathcal{H})$.*

It is enough to show that only unity commutes with our four generators. We shall work in coordinate representation and realize the arbitrary operator A as an integral one with kernel $A(x, y)$ (possibly a generalized function).

$$A\psi(x) = \int A(x, y)\psi(y)dy .$$

Operators u and \hat{v} act as shifts

$$u\psi(x) = \psi(x + q), \quad \hat{v}\psi(x) = \psi\left(x + \frac{2\pi\hbar}{p}\right),$$

and v and \hat{u} are multiplication operators

$$v\psi(x) = e^{\frac{ipx}{\hbar}}\psi(x), \quad \hat{u}\psi(x) = e^{\frac{2\pi ix}{q}}\psi(x) .$$

Commutativity of A with \hat{u} shows, that

$$A(x, y) = \sum a_n(x)\delta(x - y - nq)$$

and then commutativity with v leads to the condition

$$a_n(x)e^{\frac{ipx}{\hbar}}(e^{2\pi i\theta n} - 1) = 0.$$

If θ is irrational the last factor is not zero unless $n = 0$ and we get

$$a_n(x) = 0, \quad n \neq 0.$$

Thus A is a multiplication operator

$$A(x, y) = a(x)\delta(x - y).$$

Now commutativity with u and \hat{v} leads to condition of double periodicity of $a(x)$

$$a(x) = a(x + q),$$

$$a(x) = a\left(x + \frac{2\pi\hbar}{p}\right).$$

The first condition shows that

$$f(x) = \sum f_n e^{\frac{2\pi inx}{q}}$$

and the second gives

$$\sum f_n e^{\frac{2\pi i n x}{q}} (e^{\frac{2\pi i n}{\theta}} - 1) = 0,$$

from which due to irrationality of $1/\theta$ we see that

$$f_n = 0, \quad n \neq 0.$$

Finally we get

$$A(x, y) = f_0 \delta(x - y),$$

so that A is a multiple of unity, which proves the Lemma.

For illustration I present a proper form for operators f and g introduced above which was obtained in [3]. The answer looks as follows. Let us use P and Q as generators of \mathcal{B} . Operators f and g , defining the shift, are to be looked for as functions of P and Q respectively

$$f = r(P), \quad g = r(Q).$$

Function $r(P)$ satisfies the functional equation

$$\frac{r(P+p)}{r(P)} = e^{\frac{-iPq}{\hbar}} e^{-i\pi\theta}$$

with a solution

$$r(P) = \exp \frac{-\pi i \theta P^2}{p^2}.$$

It is not evident, that this $r(P)$ is generated by u and \hat{v} as it should. However the following is true.

Consider a more complicated functional equation

$$\frac{s(P+p)}{s(P)} = \frac{1}{1 + e^{\frac{iPq}{\hbar}}}.$$

The solution is given by the integral

$$s(P) = \exp \frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{\frac{Pq}{\hbar}\xi}}{\sinh \pi \xi \sinh \pi \theta \xi} \frac{d\xi}{\xi},$$

where the singularity at $\xi = 0$ is circled from above. The calculation of the integral by residues leads to the expression

$$s(P) = \frac{s_{\theta}(u)}{s_{\hat{\theta}}(\hat{v})}$$

where

$$\hat{\theta} = -\frac{1}{\theta}$$

and

$$s_{\theta}(u) = \exp \frac{1}{2i} \sum \frac{(-1)^n u^n}{n \sinh \pi n \theta}.$$

For irrational real θ the series in the exponent suffers from the small denominators, however the ratio $s(P)$ is well defined. We see that $s(P)$ is an element in algebra generated by u and \hat{v} .

Now we have

$$e^{\frac{-\pi i \theta P^2}{p^2}} = \text{const } s(P)s(-P)$$

Thus the operators f and g , generating shift $u \rightarrow v$ can be explicitly constructed after our extension of algebra as function of u, v, \hat{u}, \hat{v} . In other words, the shift is an outer automorphism of the algebra A_θ .

Now we turn to the first item of the title, namely the modular group. It is clear that \hat{u} and \hat{v} also constitute a Weyl pair

$$\hat{u}\hat{v} = e^{\frac{-2\pi i}{\theta}}\hat{v}\hat{u}$$

The only difference from the original algebra of u and v is the change of parameter

$$\theta \rightarrow -\frac{1}{\theta}$$

resembling one of the transformations of the modular group, usually called S . Another one, namely T

$$\theta \rightarrow \theta + 1$$

does not change the algebra \mathcal{A}_θ . Thus we have natural action of S and T on generators u, v

$$S(u) = \hat{u}, \quad S(v) = \hat{v},$$

$$T(u) = u, \quad T(v) = v.$$

It continues further as follows

$$S(\hat{u}) = u, \quad S(\hat{v}) = v,$$

$$T(\hat{u}) = \hat{u}^{-1}u, \quad T(\hat{v}) = \hat{v}^{-1}v.$$

It is easy to check, that the relations of modular group are satisfied in the following form

$$S^2 = \text{id}; \quad (TS)^3 = \sigma$$

where σ is an automorphism of A_θ

$$\sigma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^{-1} \\ v^{-1} \end{pmatrix}.$$

and the same for \hat{u} and \hat{v} . Thus we see that with any irrational θ we can associate the action of the modular group on the algebra \mathcal{B} .

Pair of algebras A_θ and $A_{-1/\theta}$ was considered by A.Connes in a different context [4]. He commented that A_θ (and $A_{-1/\theta}$) is a factor II_1 . Indeed, one can introduce a trace over A_θ in the following manner. For general element

$$a = \sum a_{mn} u^m v^n$$

we have

$$\text{tr} a = a_{00}$$

It is easy to check that

$$\text{tr}(ab) = \text{tr}(ba)$$

so that the main property of trace is satisfied. For irrational θ algebra \mathcal{A}_θ is infinite dimensional factor in \mathcal{B} . However, $tr(\mathbf{I}) = 1$. That is why it is factor II_1 .

Our assertion shows that for irrational θ factors A_θ and $A_{-1/\theta}$ are commutants of each other in \mathcal{B} and together they generate \mathcal{B} . In other words we see that one degree of freedom is virtually divided into two! Whereas this comment is irrelevant for one particle quantum mechanics, it could be important in applications to quantum field theory, where the Weyl-type operators appear in lattice regularizations.

Indeed, in such a regularization one uses the exponents of canonical fields and usually speaks of the "compactified" fields with θ being called a "radius of compactification" (see, e.g., [5]). Operators a-la u and v (or w_n , $n = 1, \dots$) serve as generators of these "compactified" degrees of freedom. The dual operators a-la \hat{u} and \hat{v} are to be interpreted as generators of additional degrees of freedom, called "winding" modes or solitonic modes etc.

Radii of compactification play the role of the coupling constants with the usual solitonic feature [6]: weak interaction of main modes corresponds to strong interaction of solitonic modes.

I believe that naive mathematics presented above bears the essential explanation of the necessity of taking into account the solitonic modes and appearance of duality and/or modular covariance in dynamical problems.

References

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