DECOMPOSING THE YANG-MILLS FIELD

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Recently we have proposed a set of variables for describing the physical parameters of $SU(N)$ Yang–Mills field. Here we propose an off-shell generalization of our Ansatz. For this we envoke the Darboux theorem to decompose arbitrary one-form with respect to some basis of one-forms. After a partial gauge fixing we identify these forms with the preimages of holomorphic and antiholomorphic forms on the coset space $SU(N)/U(1)^{N-1}$, identified as a particular coadjoint orbit. This yields an off-shell gauge fixed decomposition of the Yang-Mills connection that contains our original variables in a natural fashion.

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[*] supported by Russian Fund for Fundamental Science

[**] Supported by NFR Grant F-AA/FU 06821-308

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Recently we have proposed a novel decomposition of the four dimensional $SU(N)$ Yang-Mills connection A_μ^a [1], [2]. This decomposition introduces $r = N-1$ mutually orthogonal unit vectors m_i with r the rank of $SU(N)$, *i.e.* one for each generator of the Cartan subalgebra $U(1)^{N-1}$. In addition there are r abelian Higgs multiplets (C_μ, ρ, σ) with scalars ρ and σ that transform according to irreducible representations of $SO(N-1)$ so that

$$A_\mu^a \;=\; C_\mu^i m_i^a \;+\; f^{abc}\partial_\mu m_i^b m_i^c \;+\; \rho^{ij}\; f^{abc}\partial_\mu m_i^b m_j^c \;+\; \sigma^{ij}\; d^{abc}\partial_\mu m_i^b m_j^c. \tag{1}$$

In D dimensions (1) describes $2N^2 + (D-4)N + (2-D)$ independent variables. For $D = 4$ this coincides with $2d$ where $d = N^2 - 1$ is the dimension of $SU(N)$. This is the number of physically relevant field degrees of freedom described by a $SU(N)$ Yang-Mills connection A_μ^a. Accordingly we expect that the decomposition (1) is *on-shell* complete [1], [2].

We have suggested that the field multiplets which appear in the decomposition (1) are the appropriate order parameters for describing different phases of the Yang-Mills theory. Specifically, we have proposed that (1) identifies the configurations that are responsible for color confinement [1]. Indeed, the multiplets in (1) do support field configurations that are quite natural when modelling structures such as colored flux tubes and QCD strings [3], [4]. These and some additional aspects of our decomposition have been recently studied *e.g.* in [5].

Unfortunately, the decomposition (1) becomes insufficient at the level of a quantum theory where functional integrals are invoked. There, we need an *off-shell* generalization of (1) that describes an arbitrary connection, with full $4d$ field degrees of freedom which are subjected to d gauge fixing conditions [6]. In this Letter we shall present the appropriate gauge fixed off-shell extension of (1). This extension emerges when we first introduce a decomposition of an arbitrary connection A_μ^a in terms of $4d$ independent Darboux variables. We then impose an explicit gauge fixing condition which eliminates d of these variables. In this way we get a natural gauge fixed extension of the decomposition (1), with the correct number of $3d$ field degrees of freedom.

First we illustrate our approach on the example of $N = 2$. The $SU(2)$ version of (1) involves a three component unit vector n^a, an abelian gauge field C_μ and a single complex scalar $\rho + i\sigma$. These we combine into a $U(1)$ multiplet under $SU(2)$ gauge transformations in the direction n^a with

$$A_\mu^a \;=\; C_\mu n^a \;+\; \epsilon_{abc}\partial_\mu n^b n^c \;+\; \rho\partial_\mu n^a \;+\; \sigma\epsilon^{abc}\partial_\mu n^b n^c \tag{2}$$

or, introducing matrix notations

$$A = A_\mu^a \tau^a dx^\mu, \quad \hat{\mathbf{n}} = n^a\tau^a, \tag{3}$$

where τ^a, $a = 1, 2, 3$, are Pauli matrices,

$$A = C\hat{\mathbf{n}} - i[d\hat{\mathbf{n}}, \hat{\mathbf{n}}] + \rho d\hat{\mathbf{n}} - i\sigma[d\hat{\mathbf{n}}, \hat{\mathbf{n}}]. \tag{4}$$

itis convenient to introduce a new variable g instead of $\hat{\mathbf{n}}$ as

$$\hat{\mathbf{n}} = g\tau^3 g^{-1}. \tag{5}$$

It is clear that g is defined up to a right diagonal factor $g \to gh$, so that it belongs to the coset $SU(2)/U(1) = S^2$. We introduce the Maurer-Cartan one-forms

$$L \;=\; dg\, g^{-1} \quad , \quad R \;=\; g^{-1}dg.$$

We then have

$$d\hat{\mathbf{n}} \;=\; [L, \hat{\mathbf{n}}] \;=\; g[R, \tau^3]g^{-1}.$$

Furthermore,

$$d\hat{\mathbf{n}} \times \hat{\mathbf{n}} \;=\; \frac{1}{i}[d\hat{\mathbf{n}}, \hat{\mathbf{n}}] \;=\; \frac{1}{i}L^{off} \;=\; \frac{1}{i}gR^{off}g^{-1},$$

where *off* denotes the off-diagonal part of the matrices L and R. With this, we can write the on-shell connection (2) as follows,

$$A \;=\; g\left(C\tau^3 - \frac{1}{i}R^{diag} \;+\; \rho[R, \tau^3] \;+\; \frac{1}{i}\sigma R^{off}\right)g^{-1} \;+\; \frac{1}{i}dgg^{-1},$$

where R^{diag} is the diagonal part of R. It is manifestly gauge equivalent to the connection

$$\tilde{A} \;=\; C\tau^3 - \frac{1}{i}R^{diag} \;+\; \rho[R, \tau^3] \;+\; \frac{1}{i}\sigma R^{off} \tag{6}$$

and the transformation $g \to gh$, where $h = \exp i\alpha\tau^3$, leaves $\hat{\mathbf{n}}$ intact and clearly corresponds to an abelian gauge transformation of the $U(1)$ multiplet (C_μ, ρ, σ).

We parametrize the unit vector n^a as

$$\vec{\mathbf{n}} \;=\; \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}.$$

Then the lower off-diagonal component A^+ of A will assume the form

$$A^+ = (\rho + i\sigma)(d\theta - i\sin\theta d\phi). \tag{7}$$

We now invoke Darboux theorem to conclude that an arbitrary four dimensional one-form ϑ can be almost everywhere decomposed in terms of four functions P_i, Q^i $(i = 1, 2)$ according to

$$\vartheta_\mu dx^\mu \;=\; P_1 dQ^1 \;+\; P_2 dQ^2.$$

This is a familiar result in symplectic geometry where we commonly write the symplectic one-form in terms of canonical pairs of momenta P_i and coordinates Q^i. (In the context of a $U(1)$ gauge theory such Darboux variables have been previously used by Gliozzi [7].)

It is from this point of view that formula (7) is incomplete, it contains only half of variables necessary to parametrize the complex one-form A^+. Observe that (7) employs one-form

$$Q = d\theta - i\sin\theta d\phi$$

which is the preimage of the antiholomorphic one-form on S^2. The complete basis of one-forms on S^2 is given by Q and its complex conjugate, holomorphic one-form

$$\overline{Q} = d\theta + i\sin\theta d\phi.$$

Thus our new proposal is to complete (7) as

$$A^+ = (\rho_1 + i\sigma_1)Q + (\rho_2 + i\sigma_2)\overline{Q}. \tag{8}$$

The second off-diagonal element

$$A^- = (\rho_1 - i\sigma_1)\overline{Q} + (\rho_2 - i\sigma_2)Q \tag{9}$$

is not completely independent from A^+. It would be so if forms Q and $\overline{Q}$ in (8) and (9) were different. We propose to interpret the identity Q and $\overline{Q}$ in (8) and (9) as a partial gauge fixing. The remaining gauge freedom corresponds to abelian transformations with diagonal gauge matrices. This allows to identify C as abelian gauge field and $\Phi_1 = \rho_1 + i\sigma_1$ and $\Phi_2 = \rho_2 + i\sigma_2$ as corresponding charged scalars.

It is easy to rewrite our proposal in matrix form. One is to use the matrix R and its complex conjugate $\overline{R}$. Here is our final suggestion for the $SU(2)$ Yang–Mills field:

$$A \;=\; C\tau^3 \;+\; iR^{diag} \;+\; \rho_1[R,\tau^3] \;-\; i\sigma_1 R^{off} \;+\; \rho_2[\overline{R},\tau^3] \;+\; i\sigma_2\overline{R}^{off}. \tag{10}$$

This formula contains ten independent field components, corresponding to partial gauge fixing, cancelling two parameters out of twelve. R. Kashaev has mentioned to us that the gauge fixing can be rewritten as a gauge condition [8]

$$dA^1 \wedge A^1 \wedge A^2 \;=\; dA^2 \wedge A^2 \wedge A^1 \;=\; 0$$

The remaining $U(1)$ gauge freedom of the abelian multiplet can be fixed in any convenient manner. For example, we may select $\partial_\mu C_\mu = 0$. The ensuing fully gauge fixed connection describes $3d = 9$ independent field degrees of freedom, as it should for the gauge group $SU(2)$.

The generalization of our approach to $SU(N)$ is straightforward if we use the formulas of [2]. First, the gauge transformation analogous to one used in $SU(2)$ case is based on formula

$$m_i = g\hat{\mathbf{e}}_i g^{-1},$$

where $\hat{\mathbf{e}}_i$ denote an orthonormal basis of traceless diagonal $N \times N$ matrices with

$$\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \;=\; \frac{1}{2N}\delta_{ij} + d_{ijk}\hat{\mathbf{e}}_k$$

where d_{ijk} are completely symmetric. In terms of Maurer–Cartan matrix one-form $R = g^{-1}dg$ we construct the following one-forms

$$\begin{aligned} X_i &= [R, \hat{\mathbf{e}}_i], \\ Y_{[i,j]} &= \hat{\mathbf{e}}_i R \hat{\mathbf{e}}_j - \hat{\mathbf{e}}_j R \hat{\mathbf{e}}_i, \\ Z_{\{i,j\}} &= \hat{\mathbf{e}}_i R \hat{\mathbf{e}}_j + \hat{\mathbf{e}}_j R \hat{\mathbf{e}}_i - \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j R - R \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j. \end{aligned}$$

Notice that these differ from the corresponding formulas in [2] by $X \to gXg^{-1}$ *etc.* It is easy to see that (1) can be rewritten via those matrix forms after suitable gauge transformation as

$$A \;=\; C^i \hat{\mathbf{e}}_i - \frac{1}{i} R^{diag} + \rho_i X_i + \rho_{[i,j]} Y_{[i,j]} + \sigma_{\{i,j\}} Z_{\{i,j\}}. \tag{11}$$

One can see that only antiholomorphic forms on G/H enter into the off-diagonal elements of this connection. Thus as above we are to add also the corresponding holomorphic components using complex conjugated matrix forms $\overline{X}_i$, $\overline{Y}_{[ij]}$ and $\overline{Z}_{\{ij\}}$. The formula

$$A \;=\; C^i \hat{\mathbf{e}}_i - \frac{1}{i} R^{diag} + \rho_i X_i + \rho_{[i,j]} Y_{[i,j]} + \sigma_{\{i,j\}} Z_{\{i,j\}} + \hat{\rho}_i \bar{X}_i + \hat{\rho}_{[i,j]} \bar{Y}_{[i,j]} + \hat{\sigma}_{\{i,j\}} \bar{Z}_{\{i,j\}} \tag{12}$$

is our main proposal. It contains $3(N^2-1)+N-1$ independent parameters which exactly corresponds to the partial gauge fixing, cancelling $N^2 - N$ parameters.

In conclusion, we have presented an off-shell gauge fixed decomposition of the $SU(N)$ Yang-Mills connection, appropriate for investigating the quantum structure of the theory in terms of functional integrals. This decomposition entails both the holomorphic and antiholomorphic basis of one-forms on the maximal co-adjoint orbit $SU(N)/U(1)^{N-1}$ and generalizes our earlier on-shell construction in a natural manner. We propose that the variables which appear in our decomposition are the relevant order parameters for describing the various phases of the Yang-Mills theory. Hopefully this sheds new light to important open problems such as color confinement.

We thank R. Kashaev for most helpful discussions and J. Mickelsson for drawing our attention to [7]. L.D.F. also acknowledges the care he got in the heart clinic of the Danderyd Hospital, where same aspects of this paper were clarified.

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