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Connections of the Liouville Model and XXZ Spin Chain

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Abstract

The quantum theory of the Liouville model with imaginary field is considered using the Quantum Inverse Scattering Method. An integrable structure with nontrivial spectral parameter dependence is developed for lattice Liouville theory by scaling the L-matrix of lattice sine-Gordon theory. This L-matrix yields Bethe Ansatz equations for Liouville theory, by the methods of the algebraic Bethe Ansatz. Using the string picture of exited Bethe states, the lattice Liouville Bethe equations are mapped to the corresponding spin $\frac{1}{2}$ XXZ chain equations. The well developed theory of finite size corrections in spin chains is used to deduce the conformal properties of the lattice Liouville Bethe states. The unitary series of conformal field theories emerge for Liouville couplings of the form $\gamma = \pi\frac{\nu}{\nu+1}$, corresponding to root of unity XXZ anisotropies. The Bethe states give the full spectrum of the corresponding unitary conformal field theory, with the primary states in the Kač table parameterized by a string length K, and the remnant of the chain length mod $(\nu + 1)$.

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1 Introduction

Ever since the seminal observation by Polyakov [1] that Liouville field theory describes two-dimensional gravity as coupled to a relativistic string, much attention has been devoted to the problem of quantizing it, see e.g. [2, 3, 4, 5, 6, 7] and references therein.

Despite the fact that all the arsenals of canonical quantization [2, 3] and path integral quantization [5, 7] have been used, much remains unclear about the quantum theory, especially in the strongly coupled regime with central charge $1 < c < 25$. Some encouraging results have been acquired for this region utilizing the underlying $U_q(sl_2)$ symmetry of the quantum theory [6], and by solving the conformal Ward identities [7].

Liouville theory being a completely integrable model, the powerful methods of the Quantum Inverse Scattering Method [8] should be applicable to quantizing it. In the spirit of this method, Liouville theory was put on lattice in [4], but due to peculiarities of the integrable structure of Liouville theory, only partial quantum results were achieved. The concept of lattice Liouville theory, and the parallel concept of lattice Virasoro algebra were further developed in [9, 10, 11].

In this paper we take a somewhat different approach to putting Liouville on the lattice, more suitable for applying the methods of the algebraic Bethe Ansatz (see [12] for recent reviews). Our conventions for classical continuum Liouville theory are explained in **section 2**.

In order to use the full power of the Quantum Inverse Scattering Method, one needs a L-operator which depends non-trivially of the spectral parameter. The L-operators used in quantum Liouville theory have so far lacked this property. In **Section 3** we shall find a remedy for this shortcoming, and thus the methods of the algebraic Bethe Ansatz are in our use.

Following the line of thought of [13], we find in **Section 4** a pseudovacuum for the product of L-operators on two adjacent sites, and derive Bethe Ansatz equations for lattice Liouville theory. The equations can be regarded as Bethe Ansatz equations for a XXZ spin chain with spin $(-\frac{1}{2})$, with an extra phase factor depending on the length of the chain N. More exactly, the extra phase is related to the N:th power of the XXZ anisotropy $q = \mathrm{e}^{\mathrm{i}\gamma}$, where γ is the Liouville coupling constant.

In **Section 5** we then map the spin $(-\frac{1}{2})$ Bethe equations to the paradigmatic spin $(+\frac{1}{2})$ ones with an extra phase factor. This is done in the string approach [14] to exited states in the thermodynamic Bethe Ansatz. The mapping from Liouville to XXZ is successful only for certain root of unity anisotropies q; the Liouville coupling has to be of the form $\gamma = \pi\frac{\nu}{\nu+1}$, with ν an integer. There is a reciprocal one to one correspondence between Bethe states in the lattice Liouville model and the spin $\frac{1}{2}$ XXZ chain; highest strings are mapped to 1-strings and vice versa.

In [15, 16, 17] it was shown that conformal properties of two-dimensional statistical models at criticality can be extracted from the finite (but large) size corrections to the eigenvalues of the transfer matrix. Based on this, systematic methods for analytical calculation of the finite size corrections have been developed [18, 19, 20, 21]. For a spin chain or lattice model, the finite size analysis yields $\frac{1}{N}$ corrections to the eigenvalues of the transfer matrix, where N is the number of lattice sites.

The calculation by Karowski [19] of the conformal weights of string exited states in six-vertex and related Potts models will be of particular interest to us. In **Section 6** we review these results to for the $\frac{1}{N}$ corrections to a spin $\frac{1}{2}$ XXZ chain with extra phase factor.

Finally, in **Section 7** we interpret the results in terms of lattice Liouville theory. As the extra phase factor, $\exp\{N\pi\frac{\nu}{\nu+1}\}$ is a $\nu+1$:th root of unity, the thermodynamic limit $N \to \infty$ of our system of Bethe Ansatz equations should be taken in steps of $\nu+1$. The phase factor thus becomes a function of the remnant $\kappa = N/2 \bmod (\nu+1)$.

The finite size results show that different remnants correspond to different exited states, thus generalizing the property of spin chains that chains of even and odd length have different spectra. The scaling properties of the antiferromagnetic XXZ vacuum with $\kappa = 1$ yield the central charges of the minimal models [22] belonging to the unitary series of Friedan Qiu and Shenker [23].

For $\kappa = 0$ we get an "unrestricted" sector with central charge $c = 1$. Excluding this sector from the theory corresponds exactly to the RSOS reduction of a critical SOS model [24] . The spin $\frac{1}{2}$ Bethe equations with the extra phase are exactly the Bethe equations of a SOS model at criticality. The restriction of these models is known to produce unitary conformal field theories with $c < 1$ [25].

It is very plausible that the exclusion of the $\kappa = 0$ sector of the Hilbert space of the theory should be connected to an unitarity analysis in terms of "good" representations of the underlying $U_q(sl_2)$ symmetry of the theory, c.f. [26].

With the result of [19] that XXZ primary states emerge from a single string exited over the antiferromagnetic vacuum, we then have two integers to parameterize exited states: the string length K and the remnant κ. Working in the restricted sector, we recover from the finite size corrections the conformal weights corresponding to the whole Kač table of primary states in unitary conformal field theories, parameterized by these two integers.

We view the results announced here as encouraging when it comes to the analysis of quantum Liouville theory. Generalizing our approach to a wider class of Liouville coupling constants, possibly along the lines of [6], might shed more light on the evasive strong coupling regime.

In addition, doing the mapping of Section 5 in the inverse direction, we see how the Bethe equations of a host of critical statistical models (six-vertex, Potts, III/IV-critical RSOS) can be mapped to the lattice Liouville ones. We regard this as an explicit proof of the conformal invariance of these theories at criticality, which gives a possibility to find a Lagrangian description of the corresponding critical field theories.

2 Classical Liouville theory

We shall be quantizing Liouville theory on a Minkowskian cylinder, with the basic field $\Phi(x,t)$. The space coordinate is periodic, $x \in [0, 2\pi]$, and time is non-compact, as usual: $t \in [-\infty, \infty]$. We take the classical Liouville action in the form

$$\mathcal{S} = \frac{1}{2\gamma} \int_0^{2\pi} dx \left\{ \tfrac{1}{2}\dot{\Phi}^2 - \tfrac{1}{2}\Phi'^2 - 2\,\mathrm{e}^{-2\,\mathrm{i}\,\Phi} + 2\mathrm{i}\,\Phi'' \right\} \ . \tag{1}$$

As usual, $\dot{\Phi} \equiv \frac{\partial}{\partial t}\Phi(x,t)$ and $\Phi' \equiv \frac{\partial}{\partial x}\Phi(x,t)$. The so called conformal improvement term $\mathrm{i}\,\Phi''$ is required to make the conformal invariance manifest. It can be considered as the flat space residue of the coupling to the scalar curvature $\sqrt{g}\,R(g)\,\Phi$ of the Liouville action on a generic Riemann surface, see e.g. [5].

As it stands, the action of Equation (1) seems very non-unitary, but as we shall see in the course of this paper, and as can be inferred from [2, 3], for specific values of the coupling γ the complexities conspire to yield an unitary theory after quantization.

The equation of motion corresponding to (1) is Liouville's equation (with an imaginary field),

$$\ddot{\Phi} - \Phi'' - 4\mathrm{i}\ \mathrm{e}^{-2\,\mathrm{i}\,\Phi} = 0 \ . \tag{2}$$

In our parameterization the coupling constant does not appear in the equation of motion, it only multiplies the action. By redefining the field it is easy to recover the usual way [2, 3, 5] of having the coupling in the exponential.

To unravel the conformal invariance of the theory, it is easiest to move over to the Hamiltonian picture. Taking the conjugate momentum to be $\Pi = \frac{1}{2}\dot{\Phi}$, with Poisson Brackets

$$\{\Phi(x),\, \Pi(y)\} = -\gamma\,\delta(x-y) \ , \tag{3}$$

we get the conformally improved Hamiltonian

$$H = \int dx\ \mathcal{H} = \frac{1}{\gamma} \int dx\ \left\{ \Pi^2 + \tfrac{1}{4}\Phi'^2 + \ \mathrm{e}^{-2\,\mathrm{i}\,\Phi} \ - \mathrm{i}\,\Phi'' \right\}$$

and the improved momentum

$$P = \int dx\ \mathcal{P} = \frac{1}{\gamma} \int dx\ \left\{ \Phi'\Pi - \tfrac{\mathrm{i}}{2}\,\Pi' \right\} \ .$$

According to the prescriptions of radial quantization [22] we know that on a cylinder the role of the lightcone energy-momentum tensor is played by the sum of energy and momentum densities

$$s_+(x) = \mathcal{H}(x) + \mathcal{P}(x) \ . \tag{4}$$

This sum generates the current algebra

$$\{s_+(x),\, s_+(y)\} = 2\,(s_+(x) + s_+(y))\,\delta'(x-y) + \frac{1}{\gamma}\,\delta'''(x-y) \ .$$

Upon Fourier expanding the light-cone energy-momentum, we get a copy of the classical version of the Virasoro algebra,

$$\mathrm{i}\{L_n, L_m\} = (n-m)L_{m+n} - \frac{\pi}{2\gamma}(n^3-n)\delta_{n+m,0} \ ,$$

from which we read off the classical central charge

$$c = -6\tfrac{\pi}{\gamma} \ .$$

The difference $s_-(x) = \mathcal{H}(x) - \mathcal{P}(x)$ generates another, commuting copy of this algebra.

In [3] this system was canonically quantized using a normal ordering prescription to cope with divergences. The quantum corrections shifted the central charge to

$$c = 1 - 6(\tfrac{\pi}{\gamma} + \tfrac{\gamma}{\pi} - 2) \ ,$$

which yields minimal theories for $\frac{\gamma}{\pi}$ rational, i.e. when the deformation parameter $q = \exp i\gamma$ is a root of unity. The subset $\frac{\gamma}{\pi} = \frac{\nu}{\nu+1}, \quad \nu = 2, 3, \ldots$ corresponds to unitary theories. Similar results were obtained in [2].

3 An L-matrix for Liouville Theory

Instead of normal ordering, we shall regularize the ultraviolet divergencies by putting Liouville theory on a lattice, in a way that preserves the integrability of the model.

To get into a position where the methods of the algebraic Bethe Ansatz (see e.g. [12]) can be used, we have to find a spectral parameter dependent quantum Lax operator for lattice Liouville theory. To do this, we shall consider Liouville theory as the massless limit of sine-Gordon theory. Indeed, the sine-Gordon equation of motion[1]

$$\Box\,\Phi + 8m^2\sin(2\Phi) = 0$$

goes into the (imaginary) Liouville Equation (2), if we rescale the field:

$$\Phi \rightarrow \Phi + \mathrm{i}\,\zeta \ , \tag{5}$$

and take the limit

$$m \rightarrow 0 \ ; \qquad \zeta \rightarrow \infty \ , \quad \text{so that} \ \ m\,\mathrm{e}^{\zeta} = 1 \ . \tag{6}$$

[1]the peculiar choice of normalization of sine-Gordon mass makes the quantum group structure more transparent in the sequel.

3.1 The Sine-Gordon L-matrix

For lattice sine-Gordon there exists a Lax-operator which is based on a infinite dimensional representation of the underlying quantum group $U_q(sl_2)$ [13, 12]. In this approach, lattice sine-Gordon is treated as an inhomogeneous XXZ spin chain.

In an auxiliary matrix space a of 2×2 matrices, we define the L-operator of the n:th site of a XXZ-chain to be the matrix operator

$$L^{\mathrm{xxz}}_{n,a}(\lambda) = \begin{pmatrix} \sinh(\lambda + \mathrm{i}\gamma\, S_3^{(n)}) & \mathrm{i}\, S_-^{(n)} \sin\gamma \\ \mathrm{i}\, S_+^{(n)} \sin\gamma & \sinh(\lambda - \mathrm{i}\gamma\, S_3^{(n)}) \end{pmatrix} , \tag{7}$$

a function of the spectral parameter λ. If the quantum operators defined on the sites generate the quantum group $U_q(sl_2)$,

$$\left[S_+^{(n)},\, S_-^{(m)}\right] = \frac{\sin(2\gamma S_3^{(n)})}{\sin\gamma}\, \delta_{n,m} \; ; \qquad \left[S_3^{(n)},\, S_\pm^{(m)}\right] = \pm S_\pm^{(n)}\, \delta_{n,m} \; , \tag{8}$$

the L-operators (7) acting on auxiliary spaces a_1 and a_2 fulfill the fundamental commutation relations (FCR)

$$\begin{aligned} R_{12}(\lambda-\mu) \overset{1}{L}_n (\lambda) \overset{2}{L}_n (\mu) &= \overset{2}{L}_n (\mu) \overset{1}{L}_n (\lambda) R_{12}(\lambda-\mu) \\ \overset{1}{L}_n (\lambda) \overset{2}{L}_m (\mu) &= \overset{2}{L}_m (\mu) \overset{1}{L}_n (\lambda) \; , \text{ for } m \neq n \; . \end{aligned} \tag{9}$$

Here the usual notation for matrices on the tensor product of two auxiliary spaces is adopted; $\overset{1}{L}_n \equiv L_{n,a_1} \otimes \mathbb{1}_{a_2}$ etc.

The FCR encode the integrability of the system, and they are the basis of utilizing the algebraic Bethe Ansatz. The XXZ chain belongs to the class where the R-matrix is trigonometric:

$$R = \begin{pmatrix} \alpha & & & \\ & \beta & \delta & \\ & \delta & \beta & \\ & & & \alpha \end{pmatrix} ; \qquad \begin{array}{l} \alpha = \sinh(\lambda + \mathrm{i}\gamma) \\ \beta = \sinh\lambda \\ \delta = \mathrm{i}\,\sin\gamma \end{array} \; . \tag{10}$$

The ordered product of L-matrices around the periodic chain is the monodromy matrix

$$T(\lambda) = L_{N,a}(\lambda)\, L_{N-1,a}(\lambda) \ldots L_{1,a}(\lambda) \equiv \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix} . \tag{11}$$

The conserved quantities can now be expressed as traces (over the auxiliary space a) of powers of T, which all commute due to the fundamental commutation relations. The trace of T over the auxiliary space can be interpreted as the row-to-row transfer matrix of the corresponding two-dimensional statistical model,

$$\tau(\lambda) = \mathrm{Tr}_a(T(\lambda)) = \mathcal{A} + \mathcal{D} \; . \tag{12}$$

To get a L-matrix for sine-Gordon, we use an infinite dimensional representation of the quantum group, generated by the canonical variables Φ_n, Π_n with commutation relations

$$[\Phi_n,\ \Pi_m] = \mathrm{i}\,\gamma\,\delta_{mn}\ . \tag{13}$$

Now we can write the generators

$$S_3^{(n)} = -\tfrac{1}{\gamma}\,\Phi_n\ ; \qquad S_\pm^{(n)} = \frac{1}{2m\sin\gamma}\ \mathrm{e}^{\pm\frac{\mathrm{i}}{2}\Pi_n}\left(1+m^2\ \mathrm{e}^{2\,\mathrm{i}\,\Phi_n}\right)\mathrm{e}^{\pm\frac{\mathrm{i}}{2}\Pi_n}\ , \tag{14}$$

which fulfill the commutation relations (8).

Using Generators (14) in Equation (7) and multiplying with the matrix $-2m\,\mathrm{i}\,\sigma_1$, we get the L-matrix of lattice sine-Gordon theory

$$\begin{aligned} L_{n,a}^{SG} &= -2m\,\mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L_{n,a}^{\mathrm{xxz}} \\ &= \begin{pmatrix} h_+(\Phi_n)\ \mathrm{e}^{\mathrm{i}\,\Pi_n} & -2m\,\mathrm{i}\,\sinh(\lambda+\mathrm{i}\,\Phi_n) \\ -2m\,\mathrm{i}\,\sinh(\lambda-\mathrm{i}\,\Phi_n) & h_-(\Phi_n)\ \mathrm{e}^{-\mathrm{i}\,\Pi_n} \end{pmatrix}\ . \end{aligned} \tag{15}$$

Here we have denoted

$$h_\pm(\Phi) \equiv 1+m^2\ \mathrm{e}^{\pm 2\,\mathrm{i}\,\Phi+\mathrm{i}\,\gamma}\ .$$

Multiplying L with σ_1 is a symmetry of the FCR, so L-matrix (15) still satisfies Equation (9).

3.2 Massless Limit of the Sine-Gordon L-matrix

Now we perform the scaling and limiting procedure of Equations (5,6) on the sine-Gordon L-matrix (15), in order to get a L-matrix for lattice Liouville theory.

The functions $h_\pm$ scale to

$$h_+ \to 1\ ; \qquad h_- \to 1-\ \mathrm{e}^{-2\,\mathrm{i}\,\Phi+i\gamma} \equiv h\ .$$

Performing (5,6) directly on (15), we thus get

$$\tilde{L}_{n,a}^{\mathcal{L}} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\Pi_n} & -\ \mathrm{e}^{-\lambda-\mathrm{i}\,\Phi_n} \\ \mathrm{e}^{\lambda-\mathrm{i}\,\Phi_n} & h(\Phi_n)\ \mathrm{e}^{-\mathrm{i}\,\Pi_n} \end{pmatrix}\ . \tag{16}$$

This L-matrix with spectral parameter was acquired in [4]. The λ-dependence of $\tilde{L}_{n,a}^{\mathcal{L}}$ is trivial, though, it can be removed by a lattice gauge transformation. In other words, the quantum determinant (the central element of the algebra generated by the elements of $\tilde{L}_{n,a}$) is independent of λ. If the λ dependence is removed, we are left with the constant L-operators inherent in the approaches of [4, 6, 11]. As such, this L-matrix is not viable for the Bethe Ansatz. However, let us comment that work with Lax operator (16) leads naturally to the lattice deformation of Virasoro algebra [4, 9, 10, 11].

To get a Liouville L-matrix with non-trivial spectral parameter dependence, we have to manipulate (15) in a more involved way.[2] First, we perform a constant lattice gauge transformation on (15):

$$L^{SG}_{n,a} \to g\, L^{SG}_{n,a}\, g^{-1} \; ; \qquad g = \begin{pmatrix} m^{\frac{1}{2}} & \\ & m^{-\frac{1}{2}} \end{pmatrix}$$

Then we make the scaling (5), accompanied by a renormalization of the spectral parameter:

$$\lambda \to \lambda - \zeta \; . \tag{17}$$

After scaling and renormalizing, the limiting procedure (6) produces

$$g\, L^{SG}_{n,a}\, g^{-1} \to \begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\Pi_n} & -\mathrm{i}\;\mathrm{e}^{-\lambda - \mathrm{i}\,\Phi_n} \\ -2\mathrm{i}\,\sinh(\lambda - \mathrm{i}\,\Phi_n) & h(\Phi_n)\;\mathrm{e}^{-\mathrm{i}\,\Pi_n} \end{pmatrix} \equiv L^{\mathcal{L}}_{n,a} \; . \tag{18}$$

This is indeed the sought for quantum L-matrix for Liouville theory with a non-trivial spectral parameter, corresponding to an integrable lattice regularization of the Liouville system described by Action (1).

The easiest way to see that L-matrix (18) corresponds to an integrable lattice version of quantum Liouville theory, is to take the classical continuum limit of (18) , and find the corresponding classical continuum dynamics.

To define the classical limit, one has to recover Planck's constant. This is achieved by reinterpreting $\gamma \to \hbar\gamma$ in all quantum expressions. In the classical limit commutators turn into Poisson Brackets according to the usual Heissenberg correspondence, $\frac{\mathrm{i}}{\hbar}[,] \to \{,\}$.

Similarly, to find the continuum limit the lattice spacing a has to be recovered. The lattice mass in (15) should be $m = a\,m'$, and the *continuum* mass m' should be taken to zero according to Equation (6).

The continuum variables are defined by

$$\Pi_n \to a\,\Pi(x) \; ; \qquad \Phi_n \to \Phi(x) \; ; \qquad \delta_{mn} \to a\,\delta(x-y) \; ,$$

which maps the discrete brackets corresponding to (13) to the continuous ones of Equation (3).

Now we get in the classical continuum limit the matrix

$$U(x,\lambda) = \lim_{a,\hbar\to 0} \frac{1}{\mathrm{i}\,a}\,(L^{\mathcal{L}} - \mathbb{1}) = -\begin{pmatrix} -\Pi(x) & \mathrm{e}^{-\lambda - \mathrm{i}\,\Phi(x)} \\ 2\sinh(\lambda + \mathrm{i}\,\Phi(x)) & \Pi(x) \end{pmatrix} \; . \tag{19}$$

As the classical limit of fundamental commutation relations (9), U satisfies the so called fundamental Poisson brackets

$$\left\{ \overset{1}{U}(x,\lambda),\, \overset{2}{U}(y,\mu) \right\} = \mathrm{i}\,\gamma \left[r_{12}(\lambda-\mu),\, \overset{1}{U}(x,\lambda) + \overset{2}{U}(x,\mu) \right] \delta(x-y) \; ,$$

[2]This prescription was communicated to us by A. Volkov.

with the trigonometric classical r-matrix

$$r(\lambda) = \lim_{\hbar\to 0} \frac{-1}{\mathrm{i}\hbar} \left(\frac{R(\lambda)}{\sinh\lambda} - \mathbb{1} \right) = \frac{-1}{\sinh\lambda} \begin{pmatrix} \cosh\lambda & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \cosh\lambda \end{pmatrix} .$$

Together with the matrix

$$V(x,\lambda) = -\begin{pmatrix} \frac{1}{2}\Phi'(x) & \mathrm{e}^{-\lambda - \mathrm{i}\,\Phi(x)} \\ 2\sinh(\lambda - \mathrm{i}\,\Phi(x)) & -\frac{1}{2}\Phi'(x) \end{pmatrix} ,$$

the classical L-matrix U forms a Lax-pair for Liouville theory:

$$\dot{U} + V' + i\,[U,\,V] = 0 \iff \Box\,\Phi - 4\mathrm{i}\;\mathrm{e}^{-2\,\mathrm{i}\,\Phi} = 0 .$$

This Lax-pair can be acquired in the scaling and limiting procedure (5,17,6) from the Lax-pair of classical sine-Gordon in Reference [29].

A. Volkov brought into our attention the fact that Lax-matrix (19) is gauge equivalent to

$$\tilde{U}(x,\mu) = \begin{pmatrix} 0 & \mu - s_+ \\ 1 & 0 \end{pmatrix} ,$$

where $\mu = \exp 2\lambda$, and s_+ is the energy-momentum density of Equation (4). The Lax Equation turns into the Schrödinger equation

$$-\,\psi'' + s_+\psi = \mu\psi \;, \tag{20}$$

which usually is used in connection to the KdV equation. The role of Equation (20) for Liouville theory is stressed in [30].

4 Bethe Ansatz for lattice Liouville theory

In the algebraic Bethe Ansatz, one tries to triangularize the local L-operators in order to diagonalize the transfer matrix over the chain, which yields the conserved quantum quantities. This is done by finding a local pseudovacuum, which is annihilated by one of the off-diagonal components in $L_{n,a}$. In addition, to get an eigenstate of the transfer matrix, the pseudovacuum should be an eigenstate of the diagonal components.

The L-operator (18) developed in the previous section does not have a local pseudovacuum. However, as is the case with it's ancestor L^{SG} [13], the product of two $L^{\mathcal{L}}$:s from adjacent sites indeed has a pseudovacuum.

We denote $\Phi_{2n} = \Phi_2\,;\;\Phi_{2n-1} = \Phi_1$, and similarly for Π. The product of two Lax-operators is thus

$$\mathcal{L} = L^{\mathcal{L}}_{2n,a}\,L^{\mathcal{L}}_{2n-1,a} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{21}$$

with

$$\begin{aligned}
A &= \mathrm{e}^{\mathrm{i}(\Pi_2+\Pi_1)} - 2\,\mathrm{e}^{-\lambda-\mathrm{i}\Phi_2}\sinh(\lambda-\mathrm{i}\Phi_1)\\
B &= -\mathrm{i}\;\mathrm{e}^{-\lambda+\mathrm{i}(\Pi_2-\Phi_1)} - \mathrm{i}\,h(\Phi_1)\;\mathrm{e}^{-\lambda-\mathrm{i}(\Phi_2+\Pi_1)}\\
C &= -2\mathrm{i}\sinh(\lambda-\mathrm{i}\Phi_2)\;\mathrm{e}^{\mathrm{i}\Pi_1} - 2\mathrm{i}\,h(\Phi_2)\;\mathrm{e}^{-\mathrm{i}\Pi_2}\sinh(\lambda-\mathrm{i}\Phi_1)\\
D &= h(\Phi_2)h(\Phi_1)\;\mathrm{e}^{-\mathrm{i}(\Pi_2+\Pi_1)} - 2\sinh(\lambda-\mathrm{i}\Phi_2)\;\mathrm{e}^{-\lambda-\mathrm{i}\Phi_1}
\end{aligned}$$

Inspired by [13, 12] we make the Ansatz

$$\psi = f(\Phi_1)\,\delta(\Phi_1-\Phi_2-\gamma)$$

for the pseudovacuum at site n, with f a functional to be defined. Demanding that the off-diagonal operator C annihilates the vacuum, we get the functional relation

$$f(\Phi+\gamma) = -h(\Phi)\,f(\Phi)\ . \tag{22}$$

This equation sets very stringent conditions on the function f. It is a sufficient condition for the pseudovacuum; when it is fulfilled, the actions of A and D on ψ are diagonal, with the eigenvalues:

$$\begin{aligned}
A\,\psi &= (\,\mathrm{e}^{-2\lambda+\mathrm{i}\gamma}-1)\,\psi \equiv a(\lambda)\,\psi\\
D\,\psi &= (\,\mathrm{e}^{-2\lambda-\mathrm{i}\gamma}-1)\,\psi \equiv d(\lambda)\,\psi\ .
\end{aligned} \tag{23}$$

The treatment of Functional Relation (22) depends crucially of the value of $q = \mathrm{e}^{\mathrm{i}\gamma}$. When $|q| < 1$, Equation (22) is exactly fulfilled by the quantum dilogarithm, see Ref. [31]. Following [32], we get an explicit solution for the case $|q| = 1$ as well, which is of interest to this paper:

$$f(\Phi) = \exp\int_{-\infty}^{\infty}\frac{dx}{4x}\,\frac{\mathrm{e}^{(\gamma-\frac{\pi}{2}-\Phi)x}}{\sinh\frac{\pi}{2}x\,\sinh\frac{\gamma}{2}x}\ .$$

The singularity of the integral at $x = 0$ is left under the integration path. For more discussion on solutions of (22) and other similar functional equations we refer to [9, 10, 31, 32].

From the local pseudovacuums ψ_n we can now build up a total pseudovacuum for the quantum chain:

$$\Psi = \otimes_{n=1}^{N}\psi_n\ ,$$

where the amount of paired sites is denoted by N.

The two-site L-operators (21) become triangular when acting on the local vacuum. Thus the monodromy (11) acting on the total pseudovacuum Ψ is triangular as well:

$$T(\lambda)\,\Psi = \begin{pmatrix} a^N(\lambda)\,\Psi & * \\ 0 & d^N(\lambda)\,\Psi \end{pmatrix}\ .$$

The star in the upper right corner denotes a complicated state created by various combination of A:s, D:s and one B acting on Ψ.

Correspondingly, the transfer matrix (12) has the pseudovacuum eigenvalue

$$\tau(\lambda)\,\Psi = (a^N(\lambda) + d^N(\lambda))\,\Psi\ . \tag{24}$$

4.1 The Bethe Ansatz Equations

In the algebraic Bethe Ansatz one makes the assumption that the exited states of the theory can be obtained from the pseudovacuum by acting on it with the "pseudoparticle creation operator" $\mathcal{B}$. An arbitrary state of the form

$$\Psi_m = \prod_{j=1}^{m} \mathcal{B}(\lambda_j)\, \Psi \tag{25}$$

is characterized by m values of the spectral parameter λ_j. This state is an eigenstate of the trace of the monodromy, i.e. $\mathcal{A}+\mathcal{D}$, if the spectral parameters $\{\lambda_j\}$ satsify a set of m coupled trancendental equations known as the Bethe Ansatz equations. Written in terms of the components of R-matrix (10) and the eigenvalues a^N, d^N of $\mathcal{A}$ and $\mathcal{D}$, these equations read

$$\left[\frac{a(\lambda_k)}{d(\lambda_k)}\right]^N = \prod_{\substack{j=1 \\ j\neq k}}^{m} \frac{\alpha(\lambda_k-\lambda_j)\beta(\lambda_j-\lambda_k)}{\alpha(\lambda_j-\lambda_k)\beta(\lambda_k-\lambda_j)} \quad \forall\ k\ . \tag{26}$$

Solutions to (26) appear in sets of m spectral parameters, so called Bethe Ansatz roots, which parameterize different states of the system. A more refined analysis shows that generically all Bethe Ansatz roots are distinct, and that $m \leq N/2$, see e.g. [12].

For lattice Liouville theory, the eigenvalues a and d are given by Equation (23), and the ratio on the left hand side of Equation (26) is

$$\frac{a(\lambda)}{d(\lambda)} = \mathrm{e}^{\mathrm{i}\gamma}\frac{\sinh\left(\lambda - \mathrm{i}\frac{\gamma}{2}\right)}{\sinh\left(\lambda + \mathrm{i}\frac{\gamma}{2}\right)}\ .$$

This leads to the Bethe Ansatz equations

$$\mathrm{e}^{\mathrm{i}N\gamma}\left[\frac{\sinh\left(\lambda_k - \mathrm{i}\frac{\gamma}{2}\right)}{\sinh\left(\lambda_k + \mathrm{i}\frac{\gamma}{2}\right)}\right]^N = \prod_{j,\, j\neq k}\frac{\sinh\left(\lambda_k-\lambda_j+\mathrm{i}\gamma\right)}{\sinh\left(\lambda_k-\lambda_j-\mathrm{i}\gamma\right)} \quad \forall\ k\ . \tag{27}$$

On the other hand, the Bethe Ansatz equations of a spin-S XXZ chain are (see e.g. [12])

$$\left[\frac{\sinh\left(\lambda_k+\mathrm{i}S\gamma\right)}{\sinh\left(\lambda_k-\mathrm{i}S\gamma\right)}\right]^N = \prod_{j,\, j\neq k}\frac{\sinh\left(\lambda_k-\lambda_j+\mathrm{i}\gamma\right)}{\sinh\left(\lambda_k-\lambda_j-\mathrm{i}\gamma\right)} \quad \forall\ k\ . \tag{28}$$

Comparing (27) to (28) we see that in terms of the Bethe Ansatz, lattice Liouville theory corresponds to a spin $(-\frac{1}{2})$ XXZ spin chain, with an extra phase factor $\mathrm{e}^{\mathrm{i}N\gamma}$.

Phase factors of the form $\mathrm{e}^{\mathrm{i}\theta}$ have appeared earlier in the literature on the Bethe Ansatz. In [19] it was used to analyze Bethe Ansatz equations for Potts models, related to a seam insertion of a boundary field. Similarly, twisted boundary conditions in Reference [20] manifest themselves in the form of extra phase factors. Most interestingly, in the analysis of the eight-vertex model [33, 34], there appears a "theta-vacuum like" phase factor in the Bethe equations, related to different SOS-sectors of the theory.

Here, however, the important difference to the situations of [19, 20, 33, 34] occurs that the length N of the chain appears in the phase factor. When q is a root of unity, this will give us an extra integer parameter to parametrize "primary" exited states, in addition to the string length used in [19]. These two integers then give us enough freedom to parameterize the whole Kač table.

Thus different chain lengths occurring in the lattice Liouville model correspond exactly to the different "theta-vacuum" sectors in a critical (R)SOS model.

5 Mapping the Lattice Liouville Model to the Spin $\frac{1}{2}$ XXZ Chain

The thermodynamics [14] and finite size effects [18, 19, 20, 21] of the Bethe Ansatz of the fundamental spin $+\frac{1}{2}$ representation of XXZ-chains have been widely discussed in the literature. Accordingly it would be a desirable goal to map the spin $(-\frac{1}{2})$ Liouville Bethe equations (27) to the spin $+\frac{1}{2}$ XXZ ones (28). This can be done in the string picture [14] of solutions to the Bethe Ansatz equations.

In order to do the mapping, we thus make the string hypothesis that in the thermodynamic limit $N \to \infty$, complex Bethe Ansatz roots $\{\lambda_j\}$ cluster to complexes of the form

$$\lambda^{(K)} = \lambda_K + k\,\mathrm{i}\gamma\ , \quad k = -\tfrac{1}{2}(K-1), \dots, \tfrac{1}{2}(K-1)\ ; \qquad \mathrm{Im}\,\lambda_K \in \{0, \tfrac{\pi}{2}\}\ , \quad K \in Z_+\ . \tag{29}$$

A collection of K Bethe Ansatz roots with a common center obeying Equation (29) is known as a K-string. One-strings are the usual real Bethe Ansatz roots. Strings with $\mathrm{Im}\,\lambda_K = 0$ are called positive parity strings, if $\mathrm{Im}\,\lambda_K = \frac{\pi}{2}$ one speaks about negative parity strings.

It is important to notice that when q is a root of unity there is an upper limit to the length of the string K [14]. With

$$\left[\,\mathrm{e}^{\mathrm{i}\gamma}\right]^{\nu+1} = \pm 1\ ,$$

the maximal string length is

$$K_{\max} = \nu.$$

This limit on the range of K changes the usual concept of combinatorical completeness of Bethe Ansatz states. In Ref. [26] it was argued for a $U_q(sl_2)$ invariant spin chain at q root of unity that the Bethe Ansatz states exhibit completeness in the space of "good" representations of the quantum group.

From now on we shall concentrate only on the root of unity case. More specifically, we shall assume that the Liouville coupling constant takes only the values

$$\gamma = \frac{\pi\nu}{\nu+1}\ ; \qquad \nu \in Z_+ + 1\ .$$

Using the string picture, we shall be able to map the states of a Liouville chain with "anisotropy" γ to a spin $\frac{1}{2}$ spin chain with anisotropy

$$\tilde{\gamma} = \frac{\pi}{\nu + 1} \; ; \qquad \nu \in Z_+ + 1 \; . \tag{30}$$

We begin with positive parity strings. The number of L-strings is n_L, and the total number of Bethe Ansatz roots is

$$m = \sum_{L=1}^{\nu} L n_L \; .$$

The rapidities of different L-strings are denoted $\lambda_{L,j}$, $j = 1 \ldots n_L$.

Multiplying the Bethe Ansatz equations (27) for all the rapidities comprising a K-string, we get a set of coupled equations of the *real parts* of the strings only.

The terms in the right hand side of (26–28), which describe the effect on scattering of pseudoparticles on each other, now become scattering matrices of strings on strings.

The scattering of K-strings on 1-strings is described by the function

$$\begin{aligned} S_{K1}(\lambda) &\equiv \prod_{k=-\frac{1}{2}(K-1)}^{\frac{1}{2}(K-1)} S_{11}(\lambda) = \prod_{k=-\frac{1}{2}(K-1)}^{\frac{1}{2}(K-1)} \frac{\sinh(\lambda + \mathrm{i}k\gamma)}{\sinh(\lambda - \mathrm{i}k\gamma)} \\ &= \frac{\sinh(\lambda + \frac{1}{2}(K+1)\mathrm{i}\gamma)}{\sinh(\lambda - \frac{1}{2}(K+1)\mathrm{i}\gamma)} \frac{\sinh(\lambda + \frac{1}{2}(K-1)\mathrm{i}\gamma)}{\sinh(\lambda - \frac{1}{2}(K-1)\mathrm{i}\gamma)} \; . \end{aligned} \tag{31}$$

In terms of this function, we can write the scattering matrix of K-strings on L-strings as

$$\begin{aligned} S_{KL}(\lambda) &= \prod_{k=-\frac{1}{2}(K-1)}^{\frac{1}{2}(K-1)} \prod_{l=\frac{1}{2}(L-1)}^{\frac{1}{2}(L-1)} \frac{\sinh(\lambda + (k-l+1)\,\mathrm{i}\gamma)}{\sinh(\lambda - (k-l+1)\,\mathrm{i}\gamma)} \\ &= \prod_{k=\frac{1}{2}|K-L|}^{\frac{1}{2}(K+L)-1} S_{k1}(\lambda) = \prod_{k=\frac{1}{2}|K-L|}^{\min\left[\frac{1}{2}(K+L)-1,\; \nu-\frac{1}{2}(K+L)\right]} S_{k1}(\lambda) \; . \end{aligned} \tag{32}$$

The first expression in terms of S_{k1} bears close resemblance to (an exponential form of) the decomposition of the tensor product of two irreducible $SU(2)$ representations with spins $\frac{1}{2}(K-1)$ and $\frac{1}{2}(L-1)$. Indeed, the Bethe Ansatz can be viewed as a different way of reducing tensor products of spin 1/2 particles into irreducible representations. The factor S_{K1} attached to a spin $\frac{1}{2}(K-1)$ representation is obtained by fusing K factors S_{11} corresponding to spin 1/2 representations. The S-matrix then naturally reflects the decomposition of the tensor products of the representation spaces involved in the scattering.

The second ("reduced") expression in terms of S_{k1} is peculiar to the root of unity case. It may be related to the decomposition of the tensor product of two "good" quantum group representations, c.f. [26]. The reduced decomposition is related to the fusion rules of conformal blocks as treated in [27], suggesting that conformal blocks might be represented by Bethe Ansatz strings. In the sequel we will see, how this is to be interpreted.

The reduced form shows explicitly the symmetry of S_{KL} which will allow us to map the Liouville Bethe equations on the spin $\frac{1}{2}$ XXZ Bethe equations:

$$S_{\nu+1-K,\,\nu+1-L}(\lambda) = S_{K,L}(\lambda)\ . \tag{33}$$

Expressing the Liouville Bethe equations (27) in terms of strings we get

$$\mathrm{e}^{\mathrm{i}\,KN\gamma}\ \left[\frac{\sinh\left(\lambda_{K,i}\ -\ \mathrm{i}\,\frac{K}{2}\gamma\right)}{\sinh\left(\lambda_{K,i}\ +\ \mathrm{i}\,\frac{K}{2}\gamma\right)}\right]^N = \prod_{L;\ n_L\neq 0}\ \prod_{\substack{j=1\\(L,j)\neq(K,i)}}^{n_L} S_{KL}(\lambda_{K,i}-\lambda_{L,j}) \quad \forall\,(K,i)\ . \tag{34}$$

The amount of coupled equations is reduced to $\sum_{L=1}^{\nu} n_L$, and all roots of the equations are now taken to be real.

Similarly, applying the string picture to Equation (28) for spin $\frac{1}{2}$, we get the spin $\frac{1}{2}$ XXZ Bethe equations in terms of strings,

$$\left[\frac{\sinh\left(\lambda_{K,i}\ -\ \mathrm{i}\,\frac{K}{2}\gamma\right)}{\sinh\left(\lambda_{K,i}\ +\ \mathrm{i}\,\frac{K}{2}\gamma\right)}\right]^N = \prod_{L;\ n_L\neq 0}\ \prod_{\substack{j=1\\(L,j)\neq(K,i)}}^{n_L} \Big(S_{KL}(\lambda_{K,i}-\lambda_{L,j})\Big)^{-1} \quad \forall\,(K,i)\ . \tag{35}$$

Notice that as compared to (28), we have inverted the equation.

Inspired by Equation (33), we parameterize the string lengths from the maximal string backwards,

$$K = \nu+1-\tilde{K}\ ; \qquad L = \nu+1-\tilde{L}\ ; \qquad \tilde{K},\tilde{L} = 1,\ldots,\nu\ .$$

With this parameterization, it turns out that the Liouville Bethe Ansatz equations (34) for the strings $\{\lambda_{L,j}\}$ maps to the spin $\frac{1}{2}$ XXZ equations for a set of strings $\{\tilde{\lambda}_{\tilde{L},j}\}$ with the anisotropy $\tilde{\gamma}$.

Indeed, after some algebra we have for the left hand side

$$\mathrm{e}^{\mathrm{i}\,K\gamma}\ \frac{\sinh\left(\lambda_{K,i}\ -\ \mathrm{i}\,\frac{K}{2}\gamma\right)}{\sinh\left(\lambda_{K,i}\ +\ \mathrm{i}\,\frac{K}{2}\gamma\right)} = \mathrm{e}^{\mathrm{i}\,\tilde{K}\tilde{\gamma}}\ \frac{\sinh\left(\tilde{\lambda}_{\tilde{K},i}\ -\ \mathrm{i}\,\frac{\tilde{K}}{2}\tilde{\gamma}\right)}{\sinh\left(\tilde{\lambda}_{\tilde{K},i}\ +\ \mathrm{i}\,\frac{\tilde{K}}{2}\tilde{\gamma}\right)}\ . \tag{36}$$

The spectral parameter is changed in the following way:

$$\tilde{\lambda} = \lambda - \mathrm{i}\,(1+(-1)^K)\tfrac{\pi}{4}\ , \tag{37}$$

i.e. the parity of even-length strings is changed. In addition, it turns out that

$$S^{\gamma}_{K1}(\lambda) = \Big(S^{\tilde{\gamma}}_{\tilde{K}1}(\tilde{\lambda})\Big)^{-1}\ ,$$

where the upper index denotes whether γ or $\tilde{\gamma}$ is used when defining S_{K1} according to Equation (31).

Using this, as well as Symmetry Property (33), we get for the string-on-string scattering matrix (32)

$$S^{\gamma}_{KL}(\lambda_{K,i}-\lambda_{L,j}) = \left(S^{\tilde{\gamma}}_{\tilde{K}\tilde{L}}(\tilde{\lambda}_{\tilde{K},i}-\tilde{\lambda}_{\tilde{L},j})\right)^{-1} . \tag{38}$$

These results can easily be generalized to negative parity Liouville strings as well.

Now we are in position to state the equivalence of lattice Liouville and spin $\frac{1}{2}$ XXZ chain Bethe Ansätze. Using Equations (36, 38) in (34) and comparing to Equation (35), we get *complete equivalence* of the string states in lattice Liouville theory and a spin $\frac{1}{2}$ XXZ chain with an additional phase factor $\exp\{\mathrm{i}\,N\tilde{K}\tilde{\gamma}\}$. This extra factor will allow us to incorporate the remnant of the chain length mod $\nu+1$ as a meaningful extra parameter in the theory of finite size corrections of an usual XXZ chain.

The spin $\frac{1}{2}$ XXZ Bethe Ansatz equations we shall be analyzing are thus

$$\mathrm{e}^{\mathrm{i}\,N\tilde{K}\tilde{\gamma}} \left[\frac{\sinh\left(\tilde{\lambda}_{\tilde{K},i} - \mathrm{i}\frac{\tilde{K}}{2}\gamma\right)}{\sinh\left(\tilde{\lambda}_{\tilde{K},i} + \mathrm{i}\frac{\tilde{K}}{2}\gamma\right)}\right]^N \times \prod_{\tilde{L};\; n_{\tilde{L}}\neq 0} \prod_{\substack{j=1\\(\tilde{L},j)\neq(\tilde{K},i)}}^{n_{\tilde{L}}} S_{\tilde{K}\tilde{L}}(\lambda_{\tilde{K},i}-\lambda_{\tilde{L},j}) \;=\; 1 \quad \forall\,(\tilde{K},i)\ . \tag{39}$$

This equation is valid separately for each N, but the thermodynamic limit $N\to\infty$ is sensible only if N approaches ∞ in steps of $\nu+1$.

Accordingly, we parametrize the (even) chain length as

$$\frac{N}{2} = n(\nu+1)+\kappa\ , \quad \kappa = 0,\ldots,\nu\ . \tag{40}$$

Taking the thermodynamic limit $N\to\infty$ *at fixed* κ, letting $n\to\infty$, the limit of Bethe Ansatz equations (34, 39) is well defined. The extra phase factor in the spin $\frac{1}{2}$ XXZ chain Bethe equations (39) is thus

$$\mathrm{e}^{2\,\mathrm{i}\,\kappa\tilde{K}\tilde{\gamma}}\ . \tag{41}$$

To summarize, the obtained equivalence of string states in lattice Liouville and spin $\frac{1}{2}$ XXZ is the following:

	Lattice Liouville	Spin $\frac{1}{2}$ XXZ
anisotropy	$\gamma=\frac{\pi\nu}{\nu+1}$	$\tilde{\gamma}=\frac{\pi}{\nu+1}$
string lengths	K	$\tilde{K}=\nu+1-K$
spectral parameters	λ	$\tilde{\lambda}=\mathrm{Re}\,\lambda+\mathrm{i}\left(\mathrm{Im}\,\lambda-(1+(-1)^K)\frac{\pi}{4}\right)$

For the analysis of the physical vacuum, it is important to note that strings of maximal length are mapped to 1-strings, i.e. ordinary Bethe Ansatz roots, and vice versa.

We are intersted in the energy spectrum of the Liouville model. The auxiliary Lax-operator $L_{n,a}$ which was used for the Bethe Ansatz, intertwines the n:th quantum space and the auxiliary space sl_2. The quantum spaces for lattice Liouville model are copies of $L^2(\mathrm{R})$. Thus $L_{n,a}(\lambda)$ does not degenerate to a permutation operator at any value of the spectral parameter λ, and it is not good for investigating lattice dynamics. To get an integrable Hamiltonian that generates lattice Liouville dynamics, one would have to introduce fundamental Lax-operators $L_{n,f}$ that intertwine two quantum spaces [28].

Fortunately, we are here only interested in energy and momentum eigenvalues, so we do not have to investigate the fundamental L-operator. Following [28] one can read off the energy and momentum eigenvalues from the eigenvalues of the diagonal elements A and D of the *auxiliary* Lax-operators $L_{n,a}$. The momentum eigenvalues are

$$P^{\mathcal{L}}(\{\lambda_j\}) = \tfrac{1}{2\,\mathrm{i}} \sum_{j=1}^{m} p(\lambda_j) \; ; \qquad p(\lambda) = \ln \frac{a^{\gamma}(\lambda)}{d^{\gamma}(\lambda)} \; .$$

The upper index for a and d again stresses the particular value of anisotropy used.

The energy can be acquired by differentiating:

$$E^{\mathcal{L}}(\{\lambda_j\}) = \sum_{j=1}^{m} \epsilon\,(\lambda_j) \; ; \qquad \epsilon\,(\lambda) = \frac{\gamma}{\pi} \frac{d}{d\lambda} p(\lambda) \; .$$

From (39) we see that for the spin $\frac{1}{2}$ XXZ chain with extra phase factor, the roles of a and d are interchanged. Accordingly, using Correspondence (36), we can write the Liouville energy and momentum in terms of the XXZ ones:

$$P^{\mathcal{L}}(\{\lambda_j\}) \;=\; \tfrac{1}{\mathrm{i}} \sum_{j=1}^{m} \ln \frac{a^{\tilde{\gamma}}(\lambda_j)}{d^{\tilde{\gamma}}(\lambda_j)} = -P^{\mathrm{xxz}}(\{\lambda_j\}) \equiv \mathrm{i}\, \ln \Lambda^{\mathrm{xxz}}(\{\lambda_j\})\Big|_{\lambda=\mathrm{i}\,\frac{\tilde{\gamma}}{2}} \qquad (42)$$

$$E^{\mathcal{L}}(\{\lambda_j\}) \;=\; -E^{\mathrm{xxz}}(\{\lambda_j\}) \equiv -\mathrm{i}\,\frac{\tilde{\gamma}}{\pi}\frac{d}{d\lambda}\, \ln \Lambda^{\mathrm{xxz}}(\lambda, \{\lambda_j\})\Big|_{\lambda=\mathrm{i}\,\frac{\tilde{\gamma}}{2}} \quad . \qquad (43)$$

Here we have expressed the energy and momentum in terms of eigenvalues Λ of the transfer matrix (12),

$$\left(\mathcal{A}(\lambda) + \mathcal{D}(\lambda)\right)\, \Psi_m(\{\lambda_j\}) \equiv \Lambda(\lambda, \{\lambda_j\}) \Psi_m(\{\lambda_j\}) \; . \qquad (44)$$

This is possible for the spin $\frac{1}{2}$ XXZ chain , as the auxiliary and fundamental Lax-operators for the spin $\frac{1}{2}$ XXZ chain coincide. The XXZ Lax-operator (7) yields local commuting quantities at the value $\lambda = \mathrm{i}\,\frac{\tilde{\gamma}}{2}$, for which it becomes a permutation matrix.

The Bethe equations do not have to be completely solved to get the low lying spectrum of the Hamiltonian. We need only the *finite size* i.e. $\frac{1}{N}$ corrections to the eigenvalues of the spin $\frac{1}{2}$ XXZ transfer matrix corresponding to Bethe Ansatz equations (39). The reason for this is the following:

In this paper we started by discretizing Liouville theory in a finite volume $2\pi = aN$, with lattice spacing a. Thus the conformal scaling limit of the spin chain corresponds to the continuum limit of the Liouville field theory. Moreover, the continuum Hamiltonian is

$$H_{\rm cont} = \tfrac{1}{a} H_{\rm lattice} \ , \qquad (45)$$

so that only $\frac{1}{N}$ corrections to the eigenvalues of the lattice Hamiltonian remain finite in the continuum limit. This is exactly the realm of finite size effects, which in this case are simultaneously finite lattice spacing effects.

6 Finite Size Corrections for the six-vertex model

As is well known, the spin $\frac{1}{2}$ XXZ chain has intimate connections to the six-vertex model of classical statistical mechanics [33]. For two dimensional classical statistical models, the $\frac{1}{N}$ behaviour and the conformal properties are closely related.

In [15, 16, 17] it was argued that the central charge c of the conformal field theory corresponding to the scaling limit of a two-dimensional statistical model is related to the finite size corrections to the free energy, i.e. the logarithm of the maximal eigenvalue Λ_o of the transfer matrix, in the limit $N \to \infty$. On the other hand, the statistical mechanics minimum free energy configuration corresponds to the ground state energy E_o of the scaling conformal theory. For a model on an infinitely wide strip of length N, the behavior of E_o is [17]

$$E_o = N f_\infty - \tfrac{1}{N}\tfrac{\pi}{6}\, c + \mathcal{O}(\tfrac{1}{N^2}) \ , \qquad (46)$$

with f_∞ the free energy per site in the thermodynamic limit.

Similarly, the critical indices (conformal weights) $\Delta, \bar{\Delta}$ of operators corresponding to exited states of the system can be read off the large N behavior of configurations close to the one minimizing the free energy. In terms of the higher energy and momentum eigenvalues of the corresponding 1+1 dimensional quantum theory, critical indices are:

$$\begin{aligned} E_m - E_o &= \tfrac{2\pi}{N}(\Delta + \bar{\Delta}) \ + \ \mathcal{O}(\tfrac{1}{N^2}) \\ P_m - P_o &= \tfrac{2\pi}{N}(\Delta - \bar{\Delta}) \ + \ \mathcal{O}(\tfrac{1}{N^2}) \end{aligned} \qquad (47)$$

As opposed to our differential dependence (43), the approach of Ref. [17] relates the energy and momentum directly to lower eigenvalues Λ_m of the transfer matrix at criticality; $E_m \sim -\mathrm{Re}\ln\Lambda_m$, $P_m \sim -\mathrm{Im}\ln\Lambda_m$.

The finite size corrections of six-vertex models and the corresponding conformal properties have been extensively studied in the literature [18, 19, 20, 21]. For the six-vertex model with an extra phase factor of the form (41), and string exitations, the finite size corrections were calculated by Karowski [19]. The results of [19] relevant for us are the following.

The ground state of a six-vertex model is described by a filled Dirac sea of one-strings, i.e. $n_1 = N/2$, $n_l = 0$, $l > 1$. The logarithm of the corresponding maximal eigenvalue of the transfer matrix is [19, Eq. 4.3][3]

$$\ln \Lambda_o \approx -\mathrm{i}\, N f_\infty(\lambda) + \frac{1}{N}\frac{\pi}{6}\left(1 - \frac{6\kappa^2}{\nu(\nu+1)}\right)\cosh\left(\tfrac{\pi}{\tilde{\gamma}}\lambda\right) \ . \tag{48}$$

Exited states consist of higher strings above the vacuum, and holes in the distribution of one-strings. Low-energy excitations have both the number of holes and the number of higher strings $\sum_{L>1} n_L$ of the order N^0.

For a given distribution of strings $\{n_l\}$, there is a certain number of allowed values for the spectral parameters. This number depends on the behavior of the Bethe Ansatz equations in the limits $\lambda \to \pm\infty$. Due to the dependence of these limits on higher strings, each L-string gives automatically rise to $2L-2$ holes.

The "primary" exitations correspond to states with one higher string, no extra holes and the holes corresponding to the string evenly distibuted between the two surfaces of the Dirac sea of 1-strings, i.e. between $\lambda \sim \infty$ and $\lambda \sim -\infty$.

For such states, the finite size behavior of the eigenvalues of the transfer matrix reads[19, Eq. 4.5]

$$\ln \Lambda_m - \ln \Lambda_o \approx -\tfrac{2\pi}{N}\left\{(\Delta + \bar{\Delta})\cosh\left(\tfrac{\pi}{\tilde{\gamma}}\lambda\right) + (\Delta - \bar{\Delta})\sinh\left(\tfrac{\pi}{\tilde{\gamma}}\lambda\right)\right\} \ , \tag{49}$$

where the weights are, if the single higher string is a $\tilde{K}$-string,

$$\Delta = \bar{\Delta} = \frac{\left((\nu+1)(\tilde{K}-1) + \kappa\right)^2 - \kappa^2}{4\nu(\nu+1)} \ . \tag{50}$$

If more strings and /or holes are present in the exited state, the resulting critical indices differ from the ones above by additional integers. Accordingly, these states belong to the conformal towers of descendants of the described "primary states".

For future use we define the function

$$\delta_{L,\kappa} = \frac{\left((\nu+1)(L-1) + \kappa\right)^2}{4\nu(\nu+1)} \ ,$$

in terms of which the critical indices (50) read

$$\Delta = \bar{\Delta} = \delta_{\tilde{K},\kappa} - \delta_{1,\kappa} \ . \tag{51}$$

In this form, we explicitely see the subtraction of the part corresponding to the ground state.

[3] Note that our ν differs from the one of [19] by one. The relation between our λ and the θ of [19] is $\lambda = -\mathrm{i}\,\theta + \mathrm{i}\,\frac{\tilde{\gamma}}{2}$. This follows from the differences of the respective Bethe equations (39) and [19, Eq. 2.4].

An extra condition on $\tilde{K}$ for a primary state was found in [19]. There is an upper limit for the length of the strings that contribute to the finite size corrections. For extra phase κ, only strings satisfying

$$\tilde{K} < \nu + 1 - \kappa \tag{52}$$

contribute. This cutting off of higher strings resembles a result of Ref. [26] for a $U_q(sl_2)$ invariant XXZ chain with fixed boundary conditions. There it was conjectured that highest strings correspond to "bad" representations of the quantum group (i.e. representations with vanishing q-dimension), which have to be removed from the spectrum to keep the theory unitary.

In the six-vertex model, one expects conformal invariance at $\lambda = 0$, where the R-matrix becomes isotropic. At this point, one can use Equations (46, 47) to recognize the conformal properties of the model.

From Equation (48) we read off the central charge:

$$c = 1 - \frac{6\kappa^2}{\nu(\nu+1)} \ . \tag{53}$$

For the value $\kappa = 1$, related to critical Potts models in [19], the central charges of unitary minimal models emerge. The critical indices (50) reproduce a row in the Kač table. The highest string does not contribute to the spectrum, due to Restriction (52).

7 Conformal Properties of Lattice Liouville Theory

Now we can use the results of Ref. [19] reviewed in the previous section, to calculate the scaling properties of lattice Liouville energies and momenta, and to recognize the corresponding conformal structures.

We are interested in Bethe Ansatz states that correspond to the "primary" states of the six-vertex model, i.e. one K-string above the physical vacuum, with $K = \nu + 1 - \tilde{K}$.

From the scaling forms of the transfer matrix (49, 46) we get Liouville momenta and energies using Prescriptions (42, 43). The resulting eigenvalues yield critical indices using Equation (47).

For lattice Liouville theory, there is an important difference to the case treated in [19]. The extra phase κ is not a constant. Instead we have sectors with different values of κ, corresponding to different lengths of the chain mod $(\nu+1)$, as indicated by Equation (40). In the thermodynamic limit, all values of κ coexist.

From Equation (48) it is easy to see, that the ground state of the theory lies in the $\kappa = 0$ sector, which gives $c = 1$, the usual central charge for a periodic XXZ spin chain [18, 35].

As in [19], minimal models would emerge if the ground state was taken to be in the $\kappa = 1$ sector. Here we will adopt this approach, discarding the $\kappa = 0$ sector alltogether. Later on we will return to the interpretation of this reduction of the theory.

Accordingly, the central charge is

$$c = 1 - \frac{6}{\nu(\nu+1)} \; ; \qquad \nu = 2, 3, \ldots \tag{54}$$

and we recover the unitary minimal conformal field theories of [22, 23].

Calculating the exitation energies and the corresponding critical indices from Equation (49), we have to subtract the ground state energy, not only the minimal energy $\delta_{1,\kappa}$ within each κ sector.

With the ground state lying in the $\kappa = 1$ sector, we get for the critical indices, instead of (51),

$$\Delta = \bar{\Delta} = \delta_{\tilde{K},\kappa} - \delta_{1,1} = \frac{\Big((\tilde{K}-1)(\nu+1)+\kappa\Big)^2 - 1}{4\nu(\nu+1)} \; . \tag{55}$$

Defining

$$\begin{aligned} p &= \tilde{K} + \kappa - 1 \; ; \qquad p = 1, \ldots, \nu - 1 \\ q &= \kappa \; ; \qquad q = 1, \ldots, \nu \end{aligned} \tag{56}$$

we can write the critical indices in the form

$$\Delta = \bar{\Delta} \; = \frac{\Big(p\,(\nu+1) - q\,\nu\Big)^2 - 1}{4\nu(\nu+1)} \; . \tag{57}$$

These scaling weights reproduce the whole Kač table of unitary conformal field theories. The ranges of p and q follow accurately from Restriction (52) on the maximal string length, and the possible values of κ with the $\kappa = 0$ sector excluded.

From (45) we see that in the continuum, the energy and momentum of primary Bethe Ansatz states are

$$E = \Delta + \bar{\Delta} \; ; \qquad P = \Delta - \bar{\Delta} \; .$$

The primary Bethe states are thus products of holomorphic and antiholomorphic vectors (right and left movers) with conformal weights (55).

The behavior encountered here is exactly the same as the one encountered in the restriction of SOS models to RSOS models [24]. The Bethe Ansatz equations (39) are the Bethe equations of the so called III/IV critical limit of the SOS and corresponding eight vertex and XYZ models, in the case of root of unity anisotropy. In the III/IV critical limit, the extreme off-diagonal term (usually denoted d) in the eight-vertex R-matrix vanishes, and the elliptic functions degenerate to trigonometric ones. Thus (on applying the string picture,) the XYZ Bethe equations of [33, 34] turn into Equations (39) at criticality. For an anisotropy of the form (30), the SOS Bethe states are parameterized by all $0 \le \kappa \le \nu$. The ground state lies in the $\kappa = 0$ sector, and the corresponding central charge is $c = 1$.

In the root of unity case, it becomes possible to "restrict" the SOS model [24]. The sectors with $1 \le \kappa \le \nu$ decouple from the other sectors, and we can restrict our interest to these sectors only. This decoupled part of the SOS model is known as the RSOS model. On the level of Bethe equations, the restriction means leaving out the $\kappa = 0$ sector. The ground state now lies in the $\kappa = 1$ sector, and in the thermodynamic limit the unitary conformal field theories with $c < 1$ emerge [25].

There is one more subtlety in the interpretation of the results for the lattice Liouville model described above. Remembering the correspondence between lattice Liouville theory and the spin $\frac{1}{2}$ XXZ chain described in Section 5, it is evident that the physical vacuum for Liouville theory consists of maximal strings. This can be viewed as the maximal string limit of the fact that the vacuum of higher spin chains consists of higher strings.

Due to the complicated structure of the vacuum, not all combinations of remnant and string length are a priori allowed. On the contrary, we get stringent conditions on N from requiring the coexistence of a specific remnant $N \bmod (\nu+1)$ and a single K-string (corresponding to a spin $\frac{1}{2}$ XXZ $\tilde{K}$-string) above a sea of maximal ν-strings. In fact, these requirements fix the chain length modulo $\nu(\nu+1)$. The parameterization (40) of N has to be extended to

$$\frac{N}{2} = (n(\nu+1) - \kappa + K)\nu + K = (n\nu - \kappa + K)(\nu+1) + \kappa \; . \tag{58}$$

From here we see that for this N it is indeed possible to define a state with one K-string over a sea of $n(\nu+1) - \kappa + K$ maximal strings. In addition the remnant is κ. The thermodynamic limit has to be taken in steps of $\nu(\nu+1)$ by taking $n \to \infty$ in Equation (58).

Accordingly, the full picture of lattice Liouville primary states is the following. In a lattice Liouville chain consisting of N sites (N even), there is a primary state characterized by two integers. These integers are related to the remnants of the chain lenth $\bmod\ \nu(\nu+1)$ and $\bmod\ (\nu+1)$,

$$\frac{N}{2} \bmod \nu(\nu+1) = (\nu - p)(\nu+1) + q \; .$$

The primary state is the state with a single lower string. The $q = 0$ sector exhibits the behavior of the SOS ground state, and after the restriction, the RSOS unitary series emerge, with central charges (54) and conformal weights (57). In the thermodynamic limit all remnants mod $\nu(\nu+1)$ coexist (except possibly for the decoupled $q = 0$), and the Liouville primary states give the whole Kač table.

8 Conclusions

We have developed a spectral parameter dependent integrable structure to quantum Liouville theory on a lattice. Using the ensuing L-matrix, we have written the Bethe Ansatz equations for Liouville theory.

We have concentrated on certain Liouville coupling constants γ, for which $q = \mathrm{e}^{\mathrm{i}\gamma}$ is a root of unity. Using the string picture to describe exited Bethe Ansatz states, we have mapped the Liouville Bethe equations to a set of generalized spin $\frac{1}{2}$ XXZ Bethe Ansatz equations, more exactly the critical SOS Bethe equations. This mapping takes maximal Liouville strings to XXZ one-strings and vice versa. The physical Liouville vacuum thus consists of a Dirac sea of maximal strings.

Using results of Karowski [19] for the finite size corrections to the eigenvalues of the transfer matrix, we have calculated the central charges and conformal dimensions of the spin $\frac{1}{2}$ XXZ chain, and accordingly also of Liouville theory.

We found that the continuum limits of lattice Liouville theories with coupling constants $\gamma = \pi\frac{\nu}{\nu+1}$, $\nu = 2, 3, \ldots$ reproduce the unitary minimal models of Friedan, Qiu and Shenker [23], on restricting the chain length not to be divisible by $\nu + 1$. This restriction is the exact analogue of the RSOS restriction of SOS models at root of unity anisotropies.

Primary excitations of the Liouville chain are characterized by two integers, the length of a shorter string above the vacuum consisting of maximal strings, and the remnant κ of the chain length mod $(\nu{+}1)$. With these two parameters, the conformal weights corresponding to the exited states give all states in the corresponding Kač table.

To clarify the structure of the different sectors in the theory, a unitarity analysis based on the hidden $U_q(sl_2)$ symmetry is needed. This should illuminate both the RSOS restriction $\kappa \neq 0$, and the result of [19] that highest XXZ (i.e. lowest Liouville) strings do not contribute to the spectrum. Following [26] we believe that these properties are deeply related to properties of root of unity representations of $U_q(sl_2)$. Truncation of the Bethe Ansatz Hilbert space corresponds to excluding "bad" representations of the quantum group, which is required in order to have a positive metric on the Hilbert space.

In this work we found equivalence of the Bethe Ansatz equations of the lattice Liouville and the critical eight-vertex (SOS) models. Accordingly, the results presented here can be used to provide a Lagrangian description of the critical conformal field theories of all two-dimensional statistical models related to the eight-vertex model.

Due to the intimate connection of Liouville theory to two dimensional gravity, it would be very interesting to extend the method of quantizing Liouville theory presented here to more general couplings. It is evident that negative couplings γ correspond to the real sector of the Liouville model with $c > 25$.

Whether the strong coupling results of Gervais & al. [6] in the regime $1 < c < 25$ can be reproduced and or extended by Bethe Ansatz methods, remains to be seen. For this, one should analyze Liouville theories with imaginary couplings γ. This requires formulating the problem in terms of Baxter's equation [33], and use of quantum separation of variables in Sklyanin style [36].

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