UUITP 12/1994
hep-th/9406xxx

An involution and dynamics for the q-deformed quantum top

A.Yu.Alekseev *
Institute of Theoretical Physics, Uppsala University,
Box 803 S-75108, Uppsala, Sweden.

L.D.Faddeev
Steklov Mathematical Institute, 191011
Fontanka 27, St.Petersburg, Russia
and
Research Institute for Theoretical Physics
Siltavourenpengar 20 C, Helsinki, Finland

June 1994

Abstract

This preprint is the English translation of our paper published in Russian in the issue of St.-Petersburg journal Zapiski LOMI dedicated to the jubilee of Prof. O.A. Ladyzhenskaya in the beginning of 1993. We hope that this text still may be of interest for specialists.

It is known that the involution corresponding to the compact form is incompatible with comultiplication for quantum groups at $|q| = 1$. In this paper we consider the quantum algebra of functions on the deformed space T^*G_q which includes both the quantum group and the quantum universal enveloping algebra as subalgebras. For this extended object we construct an anti-involution which reduces to the compact form $*$-operation in the limit $q \to 1$. The algebra of functions on T^*G_q endowed with the $*$-operation may be viewed as an algebra of observables of a quantum mechanical system. The most natural interpretation for such a system is a deformation of the quantum symmetric top. We suggest a discrete dynamics for this system which imitates the free motion of the top.

*Supported by Swedish Natural Science Research Council (NFR) under the contract F-FU 06821-304; on leave of absence from Steklov Mathematical Institute, Fontanka 27, St.Petersburg, Russia

Introduction

One year ago we have introduced the quantum dynamical system T^*G_q which is a deformation of the quantum top [1]. In the nondeformed case the phase space of the symmetric top is a cotangent bundle of the group of rotations T^*G (the simplest case being $G = SU(2)$). The same phase space describes a point particle moving in the group G. After quantization the space of states $\mathcal{H}$ of the dynamical system under consideration can be naturally realized in square integrable functions on the group (in the space $L^2(G)$). The regular representation of the group G acts in the space $L^2(G)$. So, the top is a natural dynamical system closely related to representation theory.

After deformation the group G is replaced by the corresponding quantum group [2]. Parameter of the deformation plays an important role in the theory (for example, dimension of the space of states essentially depends of its value) and is denoted traditionally by q. Introduction of T^*G_q was stimulated by the problem of construction of differential calculus on quantum groups [3]-[6]. Unexpectedly, the same system appeared in the Wess-Zumino-Novikov-Witten model in conformal field theory [7]-[9]. As a result the role of quantum groups in conformal field theory was significantly clarified.

All constructions in [1] are performed in the case of complex coordinates on the group and for a complex deformation parameter. For applications the case where the group is compact and the value of q lies on the unit circle of the complex plane is the most interesting one. Thus, for $|q| = 1$ the problem of constructing an anti-involution in the algebra of observables on T^*G_q arises. An anti-involution singles out some real form of the algebra of observables. In the classical limit $q \to 1$ this real form must coincide with the algebra of observables on T^*K, where K is a compact form of the group G. Such an anti-involution will be constructed in this paper. Moreover, we make use of the case and present a dynamical system on T^*G_q which is the most precise quantum group analog of the symmetric top.

Following a common practice we shall write formulas, which can be used for any group G, but we shall illustrate them by the example of the group $SL(2)$.

In Section 1 we give a background concerning T^*G_q. The exposition follows [1]. In Section 2 we introduce and discuss an anti-involution which corresponds to the compact form of the group in the classical limit. In Section 3 we describe dynamics of the symmetric top on a quantum group.

1 Algebra of observables on T^*G_q

Let G be a classical simple Lie group, q a complex parameter, $R_+(q)$ a corresponding R-matrix [10] in the fundamental representation V. In the case of $G = SL(2)$ the fundamental representation is two-dimensional, $V = C^2$, and $R_+(q)$ is a 4×4 matrix. In the natural basis for $C^2 \times C^2$ the R-matrix is of the form

$$R_+(q) = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & q^{1/2} - q^{-3/2} & 0 \\ 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix} . \qquad (1)$$

We omit the argument q in the notation of R-matrix. Further we assume that all R-matrices are calculated for the same value of q, which enters into the definition of the theory T^*G_q as a parameter.

Along with R_+ it is useful to introduce one more $R-$matrix

$$R_- = P(R_+)^{-1}P, \qquad (2)$$

where P is a permutation in the tensor product $V \otimes V (Pa \otimes b = b \otimes a)$.

Coordinates on T^*G_q or, more precisely, generators of the algebra of functions on the noncommutative manifold T^*G_q may be combined into matrices g and $\Omega_\pm$. The matrix g is a quantum analog of a group element. Its matrix elements are coordinates on the base of the quantum bundle T^*G_q. The matrices Ω_+ and Ω_- are chosen as upper- and lower-triangular, respectively. Moreover, their diagonal parts ω_+ and ω_- are inverse to each other $\omega_+\omega_- = \omega_-\omega_+ = 1$. The entries of Ω_+ and Ω_- are coordinates in a fiber of the quantum bundle T^*G_q.

In the simplest case of $G = SL(2)$ one can write g and $\Omega_\pm$ in components

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad det_q g = ad - qbc = 1;$$

$$\Omega_+ = \begin{pmatrix} K^{1/2} & (q^{3/2} - q^{-1/2})X_+ \\ 0 & K^{-1/2} \end{pmatrix}, \Omega_- = \begin{pmatrix} K^{-1/2} & 0 \\ -(q^{1/2} - q^{-3/2})X_- & K^{1/2} \end{pmatrix} . \qquad (3)$$

The matrix elements of $\Omega_\pm$ are expressed through the generators K, X_+, X_- of the quantum algebra $sl_q(2)$ [10]. For more detail, we refer the reader to papers [1],[10]. Here we write the commutation relations for the entries of $g, \Omega_\pm$. They form a quadratic algebra, it is convenient to write this algebra in the form [1]:

$$Rg^1g^2 = g^2g^1R, \qquad (4)$$

$$\begin{aligned} R\Omega^1_+\Omega^2_+ &= \Omega^2_+\Omega^1_+R, \\ R\Omega^1_-\Omega^2_- &= \Omega^2_-\Omega^1_-R, \\ R_+\Omega^1_+\Omega^2_- &= \Omega^2_-\Omega^1_+R_+, \end{aligned} \qquad (5)$$

$$\begin{aligned} R_+\Omega^1_+g^2 &= g^2\Omega^1_+, \\ R_-\Omega^1_-g^2 &= g^2\Omega^1_-. \end{aligned} \qquad (6)$$

Here we use the notations from [10], which allow us to write the commutation relations of all entries of two matrices in a single formula. Namely, for any matrix A acting in the space V one can construct the matrices

$$A^1 = A \otimes I, \quad A^2 = I \otimes A$$

in the space $V \otimes V$. Then one can understand formulas (4-6) as relations for matrices acting in $V \otimes V$ taking into account the operator order of factors (for example, in the left-hand side of (6) the entries of $\Omega_\pm$ are situated to the left of the entries of g, and in the right-hand side to the right).

In the first three relations one can use either R_+ or R_- as the matrix R. In the fourth we must use R_+. As an alternative we write the relation

$$R_- \Omega_-^1 \Omega_+^2 = \Omega_+^2 \Omega_-^1 R_-,$$

which is obtained upon multiplication of (5) by P from the left and from the right. Here we use the relation

$$PA^1P = A^2$$

and equality (2).

It is easy to check that the form of matrices $g, \Omega_\pm$ in (3) (the matrices $\Omega_\pm$ are triangular and the q-determinant of g is equal to 1) is compatible with relations (4-6).

The subalgebras of the algebra of functions on T^*G_q generated by the entries of g and $\Omega_\pm$ respectively, are Hopf algebras. The algebra generated by the entries of g is known as the algebra of functions on the quantum group $Funk_q(G)$, and the algebra generated by the entries of $\Omega_\pm$ as the quantized universal enveloping algebra $U_q(\mathcal{G})$ where $\mathcal{G}$ is the Lie algebra corresponding to the group G. Coproduct in these Hopf algebras is given by simple formulas [10]:

$$\Delta g^{ik} = \sum_j (g^{'})^{ij} (g^{''})^{jk},$$
$$\Delta \Omega_\pm^{ik} = \sum_j (\Omega_\pm^{'})^{ij} (\Omega_\pm^{''})^{jk}. \tag{7}$$

The full algebra of functions on T^*G_q does not possess the structure of a Hopf algebra.

As is first observed in [11], it is convenient to use the matrix

$$\Omega = \Omega_+ \Omega_-^{-1}, \tag{8}$$

where Ω_-^{-1} is the antipode of the matrix Ω_-. In the case of $G = SL(2)$, the matrix Ω_-^{-1} has the following form:

$$\Omega_-^{-1} = \begin{pmatrix} K^{1/2} & 0 \\ (q^{3/2} - q^{-1/2}) X_- & K^{-1/2} \end{pmatrix}.$$

The commutation relations for $\Omega_\pm$ and g can be rewritten for Ω in the form

$$\Omega^1 (R_-)^{-1} \Omega^2 R_- = R_+^{-1} \Omega^2 R_+ \Omega^1 \tag{9}$$

and

$$R_- g^1 \Omega^2 = \Omega^2 R_+ g^1. \tag{10}$$

Let us consider the classical limit of the relations (9-10). We set

$$q = e^{i\hbar\gamma}, \tag{11}$$

where γ is a parameter of deformation and $\hbar$ is the Planck constant which controls the passage from classical to quantum mechanics. One commonly uses $\hbar = 1$, but, by methodical considerations, it is convenient to preserve $\hbar$ as an independent constant in the theory. For small γ, we can expand the R-matrices R_+ and R_- in the series with respect to γ:

$$R_\pm = I + i\hbar\gamma r_\pm + \dots \ , \tag{12}$$

where $r_\pm$ are classical $r-$matrices. For example, in the case $G = SL(2)$

$$r_+ = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 2 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix},$$
$$r_- = -P r_+ P = \begin{pmatrix} -1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & -2 & 1/2 & 0 \\ 0 & 0 & 0 & -1/2 \end{pmatrix}. \tag{13}$$

The difference of the classical r-matrices

$$C = r_+ - r_- \tag{14}$$

coincides with the tensor Casimir operator for Lie algebra $\mathcal{G}$. For example, for $\mathcal{G} = sl(2)$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \sum_{a=1}^{3} \sigma^a \otimes \sigma^a, \tag{15}$$

where σ^a are Pauli matrices.

In the classical limit $\gamma \to 0$ we set

$$\Omega = I + \gamma\omega + \dots \ . \tag{16}$$

Then, by setting $\gamma = 0$, we have the following relations for the matrices g and ω:

$$\begin{aligned} g^1 g^2 &= g^2 g^1 \ , \\ \omega^1\omega^2 - \omega^2\omega^1 &= -i\hbar[C, \omega^2] \ , \\ g^1\omega^2 - \omega^2 g^1 &= i\hbar C g^1 \ . \end{aligned} \tag{17}$$

By setting $\hbar \to 0$, we obtain the Poisson brackets for the entries of g and ω:

$$\begin{aligned} \{g^1, g^2\} &= 0, \\ \{\omega^1, \omega^2\} &= \frac{1}{2}[C, \omega^1 - \omega^2], \\ [\omega^1, g^2] &= -Cg^2. \end{aligned} \tag{18}$$

It is easy to check that the Poisson brackets (18) coincide with the canonical Poisson structure on the manifold T^*G, where g is a group element and ω is a right-invariant momentum chosen as a coordinate on the fiber.

This completes the short review of main properties of the model T^*G_q and we pass to the informative part of the paper.

2 An anti-involution for T^*G_q

We assume that q satisfies the relation

$$qq^* = 1. \tag{19}$$

In that case the $R-$matrices R_+ and R_- are Hermitian conjugate with each other

$$(R_+)^* = R_-. \tag{20}$$

It is easy to introduce an operation $*$ for the matrices $\Omega_\pm$ compatible with relations (5). Namely, we can put

$$\Omega_+^* = \Omega_-, \quad \Omega_-^* = \Omega_+. \tag{21}$$

In terms of generators of the quantum algebra we have a $q-$analog of the $SU(2)$-involution:

$$K^* = K^{-1}, \quad X_+^* = X_-, \quad X_-^* = X_+. \tag{22}$$

In the classical limit (16) formulas (21) turn into the relation

$$\omega_+^* = \omega_-, \quad \omega_-^* = \omega_+. \tag{23}$$

For the complete matrix

$$\omega = \omega_+ - \omega_-$$

we see that ω is anti-Hermitian:

$$\omega^* = -\omega. \tag{24}$$

Condition (24) singles out the compact form of the algebra $\mathcal{G}$. It becomes clear that the similar condition for the group

$$g^* = g^{-1} \tag{25}$$

is not compatible with relation (4) and it can not be used in the definition of an anti-involution for T^*G_q. The rest of the section is devoted to the description of a way out the situation and to the construction of g^*.

In order to describe an alternative to (25) we need several new objects. It is convenient to consider, together with the right-invariant momentum Ω_L generating left translations on the group, the left-invariant momentum Ω_R generating right translations. Ω_R is expressed through Ω_L by means of the simple formula

$$\Omega_R = g^{-1}\Omega_L g. \tag{26}$$

It is easy to check that Ω_R and Ω_L commute,

$$\Omega_L^1\Omega_R^2 = \Omega_R^2\Omega_L^1. \tag{27}$$

By analogy with Ω_L, we decompose Ω_R into a product of upper- and lower-triangular matrices with equal diagonal parts (we mean Σ_+ and Σ_-^{-1})

$$\Omega_R = \Sigma_+\Sigma_-^{-1}, \tag{28}$$

so that

$$g^{-1}\Omega_\pm = \Sigma_\pm h. \tag{29}$$

The coordinates $\Sigma_\pm$ and h yield an alternative parametrization on T^*G_q. In the classical limit $\gamma \to 0$ the matrices $\Omega_\pm$ and $\Sigma_\pm$ tend to 1, and we get

$$g^{-1} = h. \tag{30}$$

The required anti-involution is of the form

$$g^* = h. \tag{31}$$

It turns out to be compatible with basic relations (4-6). Observe that the components of the right momentum Ω_R

$$\Sigma_\pm = g^{-1}\Omega_\pm h^{-1} \tag{32}$$

behave in the same way as the components of the left momentum:

$$\Sigma_+^* = \Sigma_-, \quad \Sigma_-^* = \Sigma_+ \tag{33}$$

(cf. formula (21)).

In the algebra of functions on T^*G_q relations (21) and (30) determine an anti-involution compatible with relations (4-6). This allows us to say that we have constructed a q-analog of the cotangent bundle of a compact group (in the simplest case, of the group $SU(2)$) for $|q| = 1$. We observe that in this framework an analog of the compact group itself is not well defined, since the involution (25) is not compatible with relations (4). More precisely, as is easy to see, g^* and g^{-1} have different commutation relations and we can not make them equal without contradictions. Thus, the main conclusion is that, despite of lack of q-analogs of compact groups for $|q| = 1$, one can successfully construct q-analogs of their cotangent bundles, which is to a large extent an adequate alternative.

In conclusion of this section, we introduce a set of relations for $\Sigma_\pm$ and h:

$$R_\pm h^1 h^2 = h^2 h^1 R_\pm;$$

$$\begin{aligned}\Sigma^1_+\Sigma^2_+ R_\pm &= R_\pm \Sigma^2_+\Sigma^1_+,\\ \Sigma^1_-\Sigma^2_- R_\pm &= R_\pm \Sigma^2_-\Sigma^1_-,\\ \Sigma^1_+\Sigma^2_- R_+ &= R_+ \Sigma^2_-\Sigma^1_+;\end{aligned}$$

$$\begin{aligned}h^1\Sigma^2_+ &= \Sigma^2_+ R_- h^1,\\ h^1\Sigma^2_- &= \Sigma^2_- R_+ h^1,\end{aligned}$$

$$\Omega^1\Sigma^2 = \Sigma^2\Omega^1;$$

$$\begin{aligned}g^1 R_-\Sigma^2_+ &= \Sigma^2_+ g^1,\\ g^1 R_+\Sigma^2_- &= \Sigma^2_- g^1;\end{aligned}$$

$$\begin{aligned}h^1\Omega^2_+ &= \Omega^2_+ h^1 R_-,\\ h^1\Omega^2_- &= \Omega^2_- h^1 R_+.\end{aligned}$$

We left to the reader interested in this topic to verify the consistency of this system of relations and its compatibility with the basic relations (4-6).

3 Dynamics on T^*G_q.

At the beginning we recall the structure of dynamics in the nondeformed case. The classical symmetric top is described by the Lagrangian

$$L = Tr(\omega_L \dot{g} g^{-1} - \frac{1}{2}\omega_L^2) = Tr(\omega_R g^{-1}\dot{g} - \frac{1}{2}\omega_L^2), \tag{34}$$

where g is a group element , ω_L and ω_R are left and right momenta, respectively, and

$$\omega_R = g^{-1}\omega_L g. \tag{35}$$

The equations of motion are of the form

$$\dot{g} = \omega_L g, \quad \dot{\omega}_L = 0 \tag{36}$$

or

$$\dot{g} = g\omega_R, \quad \dot{\omega}_R = 0. \tag{36}$$

The solution of the equations (36) can written as follows

$$\begin{aligned}\omega_L(t) = \omega_L(0) &= \omega_L,\\ g(t) &= e^{\omega_L t} g(0)\end{aligned} \tag{37}$$

and it gives rise to the mapping $(g(0), \omega(0)) \to (g(t), \omega(t))$, preserving the Poisson structure on T^*G.

We would like to construct an analog of such a mapping for the deformed case. Such an analog does exist but one can not make the time variable t continuous in a natural way. It turns out to be discrete and takes say integer values $t = 0, 1, 2, \ldots$ and our dynamical system turns into a cascade. The values of the variables $g(t)$ and $\Omega(t)$ are given by the formulas

$$\begin{aligned} \Omega(n) &= \Omega(0), \\ g(n) &= \Omega^n g(0). \end{aligned} \tag{38}$$

The main property of the transformations (38) consists in preserving (4), (9), (10). It is sufficient to check that this property holds for $n = 1$. We make these simple calculations

$$\begin{aligned} &\Omega^1 g^1 \Omega^2 g^2 = && \text{(by formula (10))} \\ &= \Omega^1 (R_-)^{-1} \Omega^2 R_+ g^1 g^2 = && \text{(by formulas (4) and (9))} \\ &= R_+^{-1} \Omega^2 R_+ \Omega^1 R_-^{-1} g^2 g^1 R_+ = && \text{(using (10))} \\ &= R_+^{-1} \Omega^2 R_+ R_+^{-1} g^2 \Omega^1 g^1 R_+ = \\ &= R_+^{-1} \Omega^2 g^2 \Omega^1 g^1 R_+. \end{aligned}$$

which coincides with relation (4):

$$R_+ \Omega^1 g^1 \Omega^2 g^2 = \Omega^2 g^2 \Omega^1 g^1 R_+.$$

Further we consider the expression

$$\begin{aligned} &R_- \Omega^1 g^1 \Omega^2 = && \text{(using (10))} \\ &= R_- \Omega^1 R_-^{-1} \Omega^2 R_+ g^1 = && \text{(using (9))} \\ &= \Omega^2 R_+ \Omega^1 R_+^{-1} R_+ g^1 = \\ &= \Omega^2 R_+ \Omega^1 g^1, \end{aligned}$$

which reproduces the formula (10).

Thus, we have verified the basic relations for $n = 1$. By induction, the same is true for any natural n.

Observe one more important property of the evolution (38). Namely, it is compatible with the anti-involution proposed in the previous section. This means that if we start with $\Omega_\pm(0)$ and $g(0)$ such that

$$\Omega_+^*(0) = \Omega_-(0), \quad g^*(0) = h(0),$$

then $\Omega_\pm$ and g will satisfy these relations at any time $t = n$. We check this assertion. First, we have

$$\Omega_\pm(n) = \Omega_\pm(0)$$

and so the relations for $\Omega_\pm(n)$ are valid automatically.

We again check the relations for g only for $n = 1$. In that case

$$\begin{aligned} g^{-1}(1)\Omega_+ &= \Sigma_+(1) h(1), \\ g^{-1}(1)\Omega_- &= \Sigma_-(1) h(1). \end{aligned} \tag{39}$$

Using relations (8), (29) and (38), it is easy to evaluate $\Sigma_\pm(1)$ and $h(1)$. They are equal to

$$\Sigma_\pm(1) = \Sigma_\pm(0), \quad h(1) = h(0)\Omega_+^{-1}\Omega_-. \tag{40}$$

Now it remains to find the relation between $g(1)$ and $h(1)$. Substituting (38) to (40) we obtain

$$\begin{aligned} g^*(1) = g^*(0)\Omega^* = h(0)(\Omega_+\Omega_-^{-1})^* = \\ = h(0)(\Omega_-^*)^{-1}\Omega_+^* = h(0)\Omega_+^{-1}\Omega_- = h(1). \end{aligned} \tag{41}$$

Thus, the discrete evolution (38) is a natural analog of the continuous evolution (37). It preserves the commutation relations of the algebra T^*G_q and is compatible with the anti-involution that singles out the compact form.

This completes the main part of the paper and we pass to discussion of applications and perspectives.

Conclusions

This paper may have meaning for several directions connected with the quantum group theory. We discuss two opportunities which, to our opinion, are of most interest.

1. As was shown in Section 3, one can naturally introduce a discrete evolution on T^*G_q in such a way that t takes integer values. At the same time, to introduce $g(t)$ for arbitrary t, or, equally, to introduce fractional powers of Ω, does not seem to be natural. In other words, deformation is clearly related to the discreteness of parameters of one-parameter automorphism groups.

Omitting here the discussion of possible physical applications, we rewrite formula (38) for $t = 1$ in the form

$$\Omega = g(1)g(0)^{-1}. \tag{42}$$

This formula is a $q-$analog of the classical formula

$$\omega = dg\, g^{-1}$$

for the Maurer-Cartan form on the group G. Here we approach the problem of constructing of differential calculus on a quantum group. This problem is discussed by many authors [3]-[6]. It seems to us that the fact that the differential (42) is not naively infinitesimal is not understood as yet. So, in the paper [6] B.Zumino makes use of the object

$$X = \Omega - I,$$

which in the classical limit $\gamma \to 0$ after renormalization tends to ω but it might be not the most natural way in the deformed case. As one of the authors of (LDF) (for example, at the seminar "Quantum Groups" of the Euler International Mathematical Institute, in the Fall 1990) repeatedly stated, he is skeptical of the construction of a differential on a quantum group with naive Leibniz rule.

We hope to return to application of the technique that we have developed here to differential calculus on quantum groups.

2. The system T^*G_q with q being a root of unity, $q^N = 1$, is most interesting for physical applications. For example, $q = e^{2\pi i/N}$. In that case the algebra $U_q(\mathcal{G})$ has only finitely many irreducible representations for which the theory of tensor products is closely related to such an area of modern mathematical physics as rational conformal field theory (RCFT). Following the analogy with the nondeformed case one can assume that the algebra T^*G_q can be represented in the q-analog of the regular representation of $U_q(\mathcal{G})$. We may assume with good reason that for q being a root of unity the regular representation turns out to be finite dimensional. Thus, an interesting problem consists in explicit constructing of the finite dimensional $*$-representation of the algebra of functions on T^*G_q. The properties of such a representation will model the main features of RCFT on a finite dimensional example and essentially explain them [7, 8].

Note Added

Since the Russian version of this paper had been written, there is some progress in the field and we acknowledge here some results and add some references missed in the text.

1. An anti-involution of the same type as the one constructed in this paper has been earlier independently considered in [12]. This concept has been applied for general quadratic R-matrix algebras parametrized by graphs in [13].

2. Differential calculus which satisfy a modified Leibniz rule and appropriate for the A-series $G = SL(n)$ has been suggested in [14].

3. The regular representation of T^*G_q will be considered among other topics in the paper [15].

References

[1] Alekseev A.Yu. , Faddeev L.D. , Commun. Math. Phys. **141**, 1991, pp.413-422

[2] Drinfeld V. , Quantum Groups, Proc. ICM-86, Berkeley, California, USA, 1986, 1987, pp.798-820

[3] Woronowicz S.L. , Commun. Math. Phys. **122**, 1989, p.125

[4] Wess J. , Zumino B. , Nucl. Phys. B (Proc. Suppl.) **18B**, 1990, pp.302-312

[5] Manin Yu.I. , Quantum groups and non-commutative de Rahm complex, Bonn preprint MPI/91-47, 1991

[6] Zumino B. , Lecture given at Leipzig, 1991

[7] Alekseev A.Yu. , Faddeev L.D. , Semenov-Tian-Shansky M.A. , Volkov A. The unravelling of the quantum group structure in the WZNW theory, preprint CERN-TH-5981/91, 1991

[8] Alekseev A.Yu. , Faddeev L.D. , Semenov-Tian-Shansky M. A. , Hidden quantum groups inside Kac-Moody algebra, Commun. Math. Phys. **149**, 1992, p.335

[9] Faddeev L.D. , Quantum symmetry in conformal field theory by Hamiltonian methods, lectures given at Cargese, summer 1991

[10] Faddeev L.D. , Reshetikhin N.Yu. , Takhtadjan L.A. , Leningrad Math. J. , v. **1**, 1989, pp.178-206

[11] Reshetikhin N.Yu. , Semenov-Tian-Shansky M.A. , Lett. Math. Phys. **19**, 1990

[12] Mack G., Schomerus V., Nucl. Phys. **B370**(1992), p.185

[13] Alekseev A. Yu. , Grosse H. , Schomerus V. , Combinatorial quantization of the Hamiltonian Chern Simons theory, HUTMP 94-B336, Commun. Math. Phys., to appear

[14] Faddeev L.D. , Pyatov P.N. , The Differential Calculus on Quantum Linear Groups, preprint hep-th/9402070

[15] Alekseev A.Yu. , Schomerus V. , Representation theory of Chern Simons observables, in preparation