Algebraic Quantization of Integrable Models in Discrete Space-time

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ABSTRACT. Just like decent classical difference-difference systems define symplectic maps on suitable phase spaces, their counterparts with properly ordered noncommutative entries come as Heisenberg equations of motion for corresponding quantum discrete-discrete models. We observe how this idea applies to a difference-difference counterpart of the Liouville equation. We produce explicit forms of of its evolution operator for the two natural space-time coordinate systems. We discover that discrete-discrete models inherit crucial features of their continuous-time parents like locality and integrability while the new-found algebraic transparency promises a useful progress in some branches of Quantum Inverse Scattering Method.

1 Introduction

In this paper we intend to elucidate the algebraic part of our approach to the quantum integrable models in 1+1 dimensional discrete space-time, developed during last five years. We shall not give a complete survey of our publications (Faddeev and Volkov 1993,1994; Faddeev 1994; Volkov 1997a,b) because it would take too much space. We believe, that the algebraic side is most instructive and original; more analytic side will be mentioned only briefly with references to (Faddeev 1994)

Discrete space-time models (DSTM) in soliton theory have acquired a prominent role from the very advent of this part of mathematical physics. The first examples of such models were proposed by Hirota (1977) as discrete analogues of the major continuous soliton models. Subsequent development was carried on mostly by Dutch group, see (Nijhoff and Capel 1995) and references therein. The recent resurgence of interest towards DSTM is connected with several new ideas:

1. The nonlinear equations for the family of transfer matrices $T_S(\lambda)$ in the framework of the Thermodynamic Bethe Ansatz can be considered as DSTM with spin S and rapidity λ being discrete variables (Klümper and Pearce 1992; Kuniba *et al.*1994; Krichever *et al.*1996). Of course, the rapidity assumes the continuous values, but only discrete shift $\lambda \to q\lambda$ enters the equations.

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2. A. Bobenko and U. Pinkal with collaborators develop the discrete analogue of classical continuous 2-dimensional differential geometry, see (Bobenko and Pinkal 199?) and references therein.

3. Quantum version of DSTM revealed a new type of symmetry, giving the discrete analogue of current algebra and Virasoro algebra (Faddeev and Volkov 1993; Volkov 1997c). Moreover, quantum DSTM seems to be rather universal, giving both massless (conformal) and massive models in continuous limit.

For the evident methodological reason we shall illustrate our approach on a concrete example. For that we have chosen the most prominent model of physics and geometry — the Liouville model. More involved Sine-Gordon model will be touched upon only briefly. Incidentally, the latter was already a subject of our earlier publications. We shall not discuss the usual paraphernalia of integrable models such as zero-curvature representation, Lax equation and Bethe ansatz. We shall simply present the main dynamical object — the evolution operator, realizing the elementary time-shift. Its natural place inside the Algebraic Bethe Ansatz is discussed in recent lectures of one of authors (Faddeev 1996). We believe, that our explicit formulas are interesting enough as they stand so we want to present them in its clearest form, independent of original derivation.

We begin with the reminder of the classical Liouville model and its Hamiltonian interpretation. This will play the role of the starting point for the subsequent deformations: discretization of space on which hamiltonian data are given, and quantization. As a result we shall get a suitable algebra of observables. Finally the time evolution will be defined in terms of a certain automorphism of this algebra. The discrete time equations of motion produced by this automorphism will be shown to be a natural analogue of the corresponding classical equations. The integrability of the model will be confirmed by presenting an explicit set of conservation laws.

2 Classical differential equation

As the goals declared in the Introduction suggest, this time we shall consider the evergreen Liouville's Equation (LE)

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = e^{-2\varphi}$$

a Hamiltonian 1+1-dimensional field theory with x denoting the spatial coordinate and t serving as time. The Cauchy data

$$\varphi(x,t)|_{t=0} = \varphi(x) \qquad\qquad \frac{\partial\varphi}{\partial t}(x,t)|_{t=0} = \varpi(x)$$

can be equipped with the canonical Poisson bracket

$$\{\varpi(x),\varphi(y)\} = \delta(x-y) \qquad\qquad \{\varpi(x),\varpi(y)\} = \{\varphi(x),\varphi(y)\} = 0$$

so that the evolution goes the Hamiltonian way in its most familiar

$$\dot{\varphi} = \{H, \varphi\} = \varpi \qquad\qquad \dot{\varpi} = \{H, \varpi\} = \varphi'' + e^{-2\varphi},$$

the Hamiltonian being

$$H = \tfrac{1}{2} \int dx \; (\varpi^2 + (\varphi')^2 + e^{-2\varphi}).$$

The periodic boundary conditions

$$\varphi(x + 2\pi) = \varphi(x) \qquad\qquad \varpi(x + 2\pi) = \varpi(x)$$

pose no problem provided the Poisson bracket employs a 2π-periodic delta-function rather than the ordinary one.

Of the equation's specific features the Liouville's formula

$$e^{-2\varphi(x,t)} = \frac{f'(\xi)g'(\tau)}{(f(\xi) - g(\tau))^2}$$

$$x = \xi - \tau \qquad\qquad t = \xi + \tau$$

making a solution out of two arbitrary functions, is the ultimate. It is there, according to (Gervais and Neveu 1982; Faddeev and Takhtajan 1985), where the real Hamiltonian theory of LE begins. We shall not reach that high in this paper.

3 Classical difference equation

The best lattice approximation to LE

$$e^{\varphi(x,t-\Delta)} e^{\varphi(x,t+\Delta)} - e^{\varphi(x-\Delta,t)} e^{\varphi(x+\Delta,t)} = \Delta^2$$

is due to R. Hirota (1987) like virtually every decent difference-difference equation. In order to make its transformation into LE under limit $\Delta \to 0$ more obvious one may recompose it like this:

$$\sinh \tfrac{1}{2} \Big(\varphi(x, t - \Delta) + \varphi(x, t + \Delta) - \varphi(x - \Delta, t) - \varphi(x + \Delta, t) \Big)$$

$$= \Delta^2 e^{-\frac{1}{2}(\varphi(x,t-\Delta)+\varphi(x,t+\Delta)+\varphi(x-\Delta,t)+\varphi(x+\Delta,t))}.$$

Now as the mission of the lattice spacing Δ is over, it is only natural to have everything suitably rescaled

$$(x, t) \longrightarrow (\Delta x, \Delta t)$$

$$e^{\varphi} \longrightarrow \Delta e^{\varphi}$$

or just set Δ=1. Either way, the Difference Liouville Equation (DLE) takes its final form

$$e^{\varphi_{j,k+1}} e^{\varphi_{j,k-1}} - e^{\varphi_{j+1,k}} e^{\varphi_{j-1,k}} = 1$$

$$j + k \equiv 1 \pmod 2$$

where the change for subscripts manifests that the 'space-time' is now a $\mathbb{Z}^2$ lattice while the second line specifies which half of that lattice the equation will actually occupy. This half itself makes a square lattice turned by fourty five degrees with respect to the original one and twice less dense. The values of φ on a 'saw' formed by vertices with k equal either 0 or 1

$$\varphi_{j,0} = \varphi_j \qquad \text{for even } j$$

$$\varphi_{j,1} = \varphi_j \qquad \text{for odd } j$$

make a reasonable Cauchy data, that is they are just sufficient to have the whole system resolved step by step. This has everything to do with the second-order nature of the original continuum equation whose Cauchy data combine the present and a little bit of the future represented by $\varphi(x)$ and $\varpi(x)$ respectively.

Quite expectedly, there exists a Poisson bracket preserved under evolution along k-direction governed by DLE. However, it turns out more complicated than one might have wished a lattice deformation of the canonical one would be:

$$\{\varphi_i, \varphi_j\} = \varsigma(i,j)$$

with

$$\varsigma(i,j) = \begin{cases} 0 & \text{if } i - j \equiv 1 \pmod 2 \\ (-1)^{\frac{1}{2}(i+j+1)} \mathrm{sign}(i-j) & \text{otherwise} \end{cases}$$

Such is the price for the ultimate simplicity of the equation. This would be too much if we had lost the option of periodic boundary condition

$$\varphi_{j+L} = \varphi_j.$$

Fortunately, we had not. If the period is chosen properly

$$L = 2M \qquad\qquad M \equiv 1 \pmod 2$$

the bracket remains intact provided the above description of ς applies when $|i-j| \leq L$ and extends periodically

$$\varsigma(i \pm L, j) = \varsigma(i,j)$$

elsewhere. Those still insisting on an easier bracket can change the variables

$$\phi_j = \tfrac{1}{2}(\varphi_{j+1} + \varphi_{j-1})$$

and have it

$$\{\phi_i, \phi_j\} = 0 \qquad \text{if } |i-j| \neq 1$$

$$\{\phi_{j-1}, \phi_j\} = \frac{(-1)^j}{2}$$

at the expence of a busier equation

$$e^{2\phi_{j,k+1}} e^{2\phi_{j,k-1}} = (1 + e^{2\phi_{j+1,k}})(1 + e^{2\phi_{j-1,k}})$$

$$j + k \equiv 0 \pmod 2.$$

Either way, the prospect of dealing with discrete Poisson maps is hardly encouraging. That is why we choose to leave the classical equations alone and go quantum. Before we do, let us round out the classical part with a beautiful, if irrelevant for our current agenda, discrete Liouville formula:

$$e^{-2\phi_{j,k}} = e^{-\varphi_{j+1,k} - \varphi_{j-1,k}} = \frac{(f_{m+1} - f_m)(g_{n+1} - g_n)}{(f_{m+1} - g_n)(f_m - g_{n+1})}$$

$$j = m - n \qquad\qquad k = m + n + 1.$$

4 Algebra of observables

One dilemma about quantization is whether to develop it in terms of the bare φ's or stick to the variables actually entering the equation, that is the exponents

$$v_j = e^{\varphi_j} \qquad\qquad \{v_i, v_j\} = \varsigma(i, j) v_i v_j$$

The respective Heisenberg- and Weyl-style quantum algebras are

$$[\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j] = \mathrm{i}\hbar\gamma\varsigma(i, j)$$

$$\boldsymbol{\varphi}_{j+L} = \boldsymbol{\varphi}_j$$

with the usual lot in r.h.s. comprising the imaginary unit i, the Plank constant $\hbar$ and the coupling constant γ; and

$$\mathfrak{v}_i \mathfrak{v}_j = q^{\varsigma(i,j)} \mathfrak{v}_j \mathfrak{v}_i$$

$$\mathfrak{v}_{j+L} = \mathfrak{v}_j$$

with all packed in a single *q*uantisation constant

$$q = e^{\mathrm{i}\hbar\gamma}.$$

Weyl-style algebra may be viewed as a subalgebra of the Heisenberg-one

$$\mathfrak{v}_j = e^{\boldsymbol{\varphi}_j}$$

but not the other way round. Roughly speaking, the latter also accomodates non-integer powers

$$\mathfrak{v}_j^{\alpha} = e^{\alpha\boldsymbol{\varphi}_j}$$

not allowed in the former.

Another dilemma is whether to place q on the unit circle or not. The first option is obviously involution-friendly. It allows for both unitary

$$\mathfrak{v}_i^* = -\mathfrak{v}_i^{-1}$$

and selfadjoint

$$\mathfrak{v}_i^* = \mathfrak{v}_i$$

pictures, the former offering the luxury of dealing with bounded operators if at the expence of complications in representation theory due to the arithmetics of q while the latter actually being the one relevant for the true Liouville model. On the other hand, q inside (or outside) the circle is favoured in q-algebra but whether it is good for something else remains to be seen.

We choose not to take sides before time and conclude the Section on a more practical note. Let us introduce, for future use, quantum counterparts of the $e^{2\phi}$-variables

$$\mathfrak{w}_j = \mathfrak{v}_{j+1}\mathfrak{v}_{j-1} = \mathfrak{v}_{j-1}\mathfrak{v}_{j+1}$$

and compile a list of emerging commutation relations

$$\begin{aligned}
&\mathfrak{v}_j\mathfrak{w}_j = q^{2(-1)^j}\mathfrak{w}_j\mathfrak{v}_j \\
&\mathfrak{w}_{j-1}\mathfrak{w}_j = q^{2(-1)^j}\mathfrak{w}_j\mathfrak{w}_{j-1} \\
&\mathfrak{v}_i\mathfrak{w}_j = \mathfrak{w}_j\mathfrak{v}_i \qquad \text{if } i \neq j \pmod{L} \\
&\mathfrak{w}_i\mathfrak{w}_j = \mathfrak{w}_j\mathfrak{w}_i \qquad \text{if } |i-j| \neq 1 \pmod{L}.
\end{aligned}$$

5 Evolution operator

Given an invertible operator $\mathfrak{Q}$, one can make the algebra of observables 'evolve'

$$\cdots \longmapsto \mathfrak{Q}\mathfrak{z}\mathfrak{Q}^{-1} \longmapsto \mathfrak{z} \longmapsto \mathfrak{Q}^{-1}\mathfrak{z}\mathfrak{Q} \longmapsto \mathfrak{Q}^{-2}\mathfrak{z}\mathfrak{Q}^2 \longmapsto \cdots$$

hoping that the evolution of generators

$$\mathfrak{v}_{j,k+2} = \mathfrak{Q}^{-1}\mathfrak{v}_{j,k}\mathfrak{Q} \qquad j+k \equiv 0 \pmod 2$$

$$\mathfrak{v}_{2a,0} = \mathfrak{v}_{2a} \qquad\qquad \mathfrak{v}_{2a-1,1} = \mathfrak{v}_{2a-1}$$

manages to solve some nice and local equations, for instance,

$$\mathfrak{v}_{j,k+1}\mathfrak{v}_{j,k-1} - q^{-1}\mathfrak{v}_{j-1,k}\mathfrak{v}_{j+1,k} = 1.$$

As a matter of fact, this is exactly what happens if

$$\mathfrak{Q} = \prod_{a=1}^{M} \epsilon(\mathfrak{w}_{2a-1}) \ \mathfrak{F} \prod_{a=1}^{M} \epsilon(\mathfrak{w}_{2a})$$

provided $\mathfrak{F}$ is the 'flip' operator

$$\mathfrak{F}^{-1}\mathfrak{v}_j\mathfrak{F} = \mathfrak{v}_j^{-1}$$

while the function ϵ solves the following functional equation:

$$\frac{\epsilon(qz)}{\epsilon(q^{-1}z)} = \frac{1}{1+z}.$$

Indeed, let us plug the definition of the $\mathfrak{v}_{j,k}$'s into the hypothetical equation

$$\mathfrak{Q}^{-b-1}\mathfrak{v}_{2a}\mathfrak{Q}^{b+1}\mathfrak{Q}^{-b}\mathfrak{v}_{2a}\mathfrak{Q}^{b} - q^{-1}\mathfrak{Q}^{-b}\mathfrak{v}_{2a-1}\mathfrak{Q}^{b}\mathfrak{Q}^{-b}\mathfrak{v}_{2a+1}\mathfrak{Q}^{b} = 1$$

$$\mathfrak{Q}^{-b}\mathfrak{v}_{2a-1}\mathfrak{Q}^{b}\mathfrak{Q}^{-b+1}\mathfrak{v}_{2a-1}\mathfrak{Q}^{b-1} - q^{-1}\mathfrak{Q}^{-b}\mathfrak{v}_{2a-2}\mathfrak{Q}^{b}\mathfrak{Q}^{-b}\mathfrak{v}_{2a}\mathfrak{Q}^{b} = 1$$

and dispose of as many $\mathfrak{Q}$'s as possible:

$$\mathfrak{v}_{2a}\mathfrak{Q}\mathfrak{v}_{2a} - q^{-1}\mathfrak{Q}\mathfrak{v}_{2a-1}\mathfrak{v}_{2a+1} = \mathfrak{Q}$$

$$\mathfrak{v}_{2a-1}\mathfrak{Q}\mathfrak{v}_{2a-1} - q^{-1}\mathfrak{v}_{2a-2}\mathfrak{v}_{2a}\mathfrak{Q} = \mathfrak{Q} \qquad .$$

Then all the $\epsilon(\mathfrak{w})$ factors but one go the same way which results in

$$\begin{aligned} \mathfrak{v}_{2a}\mathfrak{F}\epsilon(\mathfrak{w}_{2a})\mathfrak{v}_{2a} - q^{-1}\mathfrak{F}\epsilon(\mathfrak{w}_{2a})\mathfrak{v}_{2a-1}\mathfrak{v}_{2a+1} &= \mathfrak{F}\epsilon(\mathfrak{w}_{2a}) \\ \mathfrak{v}_{2a-1}\epsilon(\mathfrak{w}_{2a-1})\mathfrak{F}\mathfrak{v}_{2a-1} - q^{-1}\mathfrak{v}_{2a-2}\mathfrak{v}_{2a}\epsilon(\mathfrak{w}_{2a-1})\mathfrak{F} &= \epsilon(\mathfrak{w}_{2a-1})\mathfrak{F} \qquad . \end{aligned}$$

Once $\mathfrak{F}$ is gone too, we are left with

$$\begin{aligned}
\mathfrak{v}_{2a}^{-1}\epsilon(\mathfrak{w}_{2a})\mathfrak{v}_{2a} - q^{-1}\epsilon(\mathfrak{w}_{2a})\mathfrak{v}_{2a-1}\mathfrak{v}_{2a+1} &= \epsilon(\mathfrak{w}_{2a}) \\
\mathfrak{v}_{2a-1}\epsilon(\mathfrak{w}_{2a-1})\mathfrak{v}_{2a-1}^{-1} - q^{-1}\mathfrak{v}_{2a-2}\mathfrak{v}_{2a}\epsilon(\mathfrak{w}_{2a-1}) &= \epsilon(\mathfrak{w}_{2a-1})
\end{aligned}$$

which is nothing but the above functional equation mated with the commutation relations which closed the last Section:

$$\mathfrak{v}_{2a}^{-1}\epsilon(\mathfrak{w}_{2a})\mathfrak{v}_{2a} = \epsilon(q^{-2}\mathfrak{w}_{2a}) \qquad \mathfrak{v}_{2a-1}\epsilon(\mathfrak{w}_{2a-1})\mathfrak{v}_{2a-1}^{-1} = \epsilon(q^{-2}\mathfrak{w}_{2a-1})$$

$$\epsilon(q^{-2}\mathfrak{w}_j) - q^{-1}\epsilon(\mathfrak{w}_j)\mathfrak{w}_j = \epsilon(q^{-2}\mathfrak{w}_j) - q^{-1}\mathfrak{w}_j\epsilon(\mathfrak{w}_j) = \epsilon(\mathfrak{w}_j) \quad .$$

So, since everything eventually reduces to that functional equation, let us see if it can be solved.

6 q-exponent

Indeed, the equation in question

$$\frac{\epsilon(qz)}{\epsilon(q^{-1}z)} = \frac{1}{1+z}$$

is readily fulfilled by those ubiquitous q-exponents

$$\epsilon(z) = (-qz;q^2)_\infty \qquad \text{- good for } |q|<1$$

$$\epsilon(z) = \frac{1}{(-q^{-1}z;q^{-2})_\infty} \qquad \text{- good for } |q|>1$$

where

$$(x;y)_\infty \equiv \prod_{p=0}^{\infty}(1-xy^p).$$

There is no solution entire in z if $|q|=1$. For those opted for the Weyl-style algebra of observables (see Section 3) this is the end to the story, not a happy one in the latter case where the equations of motion survive but the evolution automorphism behind them turns outer.

The Heisenberg Way has more solutions to its disposal:

$$\epsilon_{\text{Heisenberg}}(z) = \epsilon_{\text{Weyl}}(z) \times \text{any function}(z^\theta)$$

with θ being the first solution

$$\theta = \frac{\pi}{\hbar\gamma}$$

to the equation

$$q^{2\theta} = 1$$

with a clear purpose:

$$\mathfrak{v}_i^\theta \mathfrak{w}_j = \mathfrak{w}_j \mathfrak{v}_i^\theta \qquad\qquad \mathfrak{v}_i \mathfrak{w}_j^\theta = \mathfrak{w}_j^\theta \mathfrak{v}_i.$$

Among them we find the one capable of surviving under $|q| \to 1$ limit (Faddeev 1994):

$$\boldsymbol{\epsilon}_q(z) = \boldsymbol{\epsilon}_{q^{\theta^2}}(z^\theta) = \begin{cases} \dfrac{(-qz;q^2)_\infty}{(-q^{-\theta^2}z^\theta;q^{-2\theta^2})_\infty} & |q|<1 \\ \\ \dfrac{(-q^{\theta^2}z^\theta;q^{2\theta^2})_\infty}{(-q^{-1}z;q^{-2})_\infty} & |q|>1. \end{cases}$$

It is plain to see what makes it so special. *Duality* is the word. We get two functional equations

$$\frac{\boldsymbol{\epsilon}_q(qz)}{\boldsymbol{\epsilon}_q(q^{-1}z)} = \frac{1}{1+z} \qquad\qquad \frac{\boldsymbol{\epsilon}_{q^{\theta^2}}(q^{\theta^2}z^\theta)}{\boldsymbol{\epsilon}_{q^{\theta^2}}(q^{-\theta^2}z^\theta)} = \frac{1}{1+z^\theta}$$

satisfied at once. Consequently, the updated evolution operator

$$\mathfrak{Q} = \prod_{a=1}^{M} \boldsymbol{\epsilon}_q(\mathfrak{w}_{2a-1})\ \mathfrak{F} \prod_{a=1}^{M} \boldsymbol{\epsilon}_q(\mathfrak{w}_{2a}) = \prod_{a=1}^{M} \boldsymbol{\epsilon}_{q^{\theta^2}}(\mathfrak{w}_{2a-1}^\theta)\ \mathfrak{F} \prod_{a=1}^{M} \boldsymbol{\epsilon}_{q^{\theta^2}}(\mathfrak{w}_{2a}^\theta)$$

$$\mathfrak{F}^{-1}\mathfrak{v}_j\mathfrak{F} = \mathfrak{v}_j^{-1} \qquad\qquad \mathfrak{F}^{-1}\mathfrak{v}_j^\theta\mathfrak{F} = \mathfrak{v}_j^{-\theta}$$

not only inherits the right evolution of the pure $\mathfrak{v}$'s

$$\mathfrak{v}_{j,k+1}\mathfrak{v}_{j,k-1} - q^{-1}\mathfrak{v}_{j-1,k}\mathfrak{v}_{j+1,k} = 1$$

but also produces an equally right evolution

$$\mathfrak{v}_{j,k+1}^\theta\mathfrak{v}_{j,k-1}^\theta - q^{-\theta^2}\mathfrak{v}_{j-1,k}^\theta\mathfrak{v}_{j+1,k}^\theta = 1$$

of their dual twins $\mathfrak{v}^\theta$. Since the Heisenberg-setup turned out to be just a pair of decoupled Weyl-ones, we will stick to the latter for the rest of the paper.

7 A different angle

Although the ingredients used in the formula of the evolution operator are all more or less familiar, they are put together in a bizarre way. A traditional R-matrix philosophy would offer a different approach which we now start presenting. The equation remains the same

$$\mathsf{v}_{m+1,n+1}\mathsf{v}_{m,n} - q^{-1}\mathsf{v}_{m,n+1}\mathsf{v}_{m+1,n} = 1$$

but the change for a kind of 'light-cone' variables

$$j = m - n \qquad\qquad k = m + n$$

signals that the n-direction is now considered temporal. So, we are going to find out what algebra of observables and what evolution operator produce the evolution

$$\mathsf{v}_{m,n+1} = \mathsf{Q}^{-1}\mathsf{v}_{m,n}\mathsf{Q}$$

$$\mathsf{v}_{m,0} = \mathsf{v}_m$$

solving that 'light-cone' equation.

8 Algebra of observables ii

Here is the complete list of relations defining the new algebra of observables:

$$\mathsf{v}_\ell\mathsf{v}_m = \mathsf{v}_m\mathsf{v}_\ell \qquad \text{if } m - \ell = 0, 2, \ldots, M-1$$

$$\mathsf{v}_\ell\mathsf{v}_m = q\mathsf{v}_m\mathsf{v}_\ell \qquad \text{if } m = 1, 3, \ldots, M$$

$$\mathsf{v}_m\mathsf{C} = q\mathsf{C}\mathsf{v}_m$$

$$\mathsf{v}_{m+M} = q^{\frac{1}{2}}\mathsf{C}\mathsf{v}_m \qquad\qquad M \equiv 1 \pmod 2.$$

This is exactly what it takes to achieve the required relationship

$$\mathsf{v}_m\mathsf{w}_m = q^2\mathsf{w}_m\mathsf{v}_m$$

$$\mathsf{v}_\ell\mathsf{w}_m = \mathsf{w}_m\mathsf{v}_\ell \qquad \text{if } \ell \neq m \pmod M)$$

between the v's and their 'derivatives'

$$\mathsf{w}_m = \frac{\mathsf{v}_{m+1}}{\mathsf{v}_{m-1}}$$

which themselves form the much advertized by the authors 'lattice current algebra'

$$\mathsf{w}_{m-1}\mathsf{w}_m = q^2\mathsf{w}_m\mathsf{w}_{m-1}$$

$$\mathsf{w}_\ell\mathsf{w}_m = \mathsf{w}_m\mathsf{w}_\ell \qquad \text{if } |m-\ell| \neq 1 \pmod{M}$$

$$\mathsf{w}_{m+M} = \mathsf{w}_m.$$

We already met similar relations, it was the end of Section 3. That time they did not contradict the periodicity of the $\mathfrak{v}$'s. Now they do: it is impossible to have $\mathsf{C} = 1$ and good commutation relations at the same time. We shall soon see why.

9 Evolution operator ii

Let us see what the operator

$$\mathsf{Q}_{\text{naive}} = \mathsf{SF}\epsilon(\mathsf{w}_M)\ldots\epsilon(\mathsf{w}_2)\epsilon(\mathsf{w}_1)$$

can do. Of course, the function ϵ and the flip operator F are the same as before

$$\frac{\epsilon(qz)}{\epsilon(q^{-1}z)} = \frac{1}{1+z}$$

$$\mathsf{F}^{-1}\mathsf{v}_m\mathsf{F} = \mathsf{v}_m^{-1}$$

while S is the shift operator

$$\mathsf{S}^{-1}\mathsf{v}_m\mathsf{S} = \mathsf{v}_{m-1}$$

We plug the 'naive' evolution into the hypothetical equation:

$$\mathsf{Q}_{\text{naive}}^{-n-1}\mathsf{v}_{m+1}\mathsf{Q}_{\text{naive}}^{n+1}\mathsf{Q}_{\text{naive}}^{-n}\mathsf{v}_m\mathsf{Q}_{\text{naive}}^{n} - q^{-1}\mathsf{Q}_{\text{naive}}^{-n-1}\mathsf{v}_m\mathsf{Q}_{\text{naive}}^{n+1}\mathsf{Q}_{\text{naive}}^{-n}\mathsf{v}_{m+1}\mathsf{Q}_{\text{naive}}^{n} = 1$$

and dispose of as many $\mathsf{Q}_{\text{naive}}$'s as possible:

$$\mathsf{v}_{m+1}\mathsf{Q}_{\text{naive}}\mathsf{v}_m - q^{-1}\mathsf{v}_m\mathsf{Q}_{\text{naive}}\mathsf{v}_{m+1} = \mathsf{Q}_{\text{naive}}.$$

Then all the $\epsilon(\mathsf{w})$ factors but one go the same way which results in

$$\mathsf{v}_{m+1}\mathsf{SF}\epsilon(\mathsf{w}_m)\mathsf{v}_m - q^{-1}\mathsf{v}_m\mathsf{SF}\epsilon(\mathsf{w}_m)\mathsf{v}_{m+1} = \mathsf{SF}\epsilon(\mathsf{w}_m).$$

Once S and F are gone too, we are left with

$$\mathsf{v}_m^{-1}\epsilon(\mathsf{w}_m)\mathsf{v}_m - q^{-1}\mathsf{v}_{m-1}^{-1}\epsilon(\mathsf{w}_m)\mathsf{v}_{m+1} = \epsilon(\mathsf{w}_m)$$

which is nothing but our functional equation mated with the commutation relations which closed the last Section:

$$\mathsf{v}_m^{-1}\epsilon(\mathsf{w}_m)\mathsf{v}_m = \epsilon(q^{-2}\mathsf{w}_m)$$
$$\epsilon(q^{-2}\mathsf{w}_m) - q^{-1}\epsilon(\mathsf{w}_m)\mathsf{w}_m = \epsilon(\mathsf{w}_m).$$

This proves that the 'naive' evolution satisfies the required equations of motion ... as long as m is neither 1 nor M. We could not reasonably expect it to do any better because $\mathsf{Q}_{\mathrm{naive}}$ obviously had no respect to the translational symmetry of the algebra of observables. In order to have this eventually repaired, let us first figure out how that sad dependence on the starting point can be cured in a simpler situation, say, for a monomial $\mathsf{w}_M^{p_M}\ldots\mathsf{w}_2^{p_2}\mathsf{w}_1^{p_1}$. Pulling the factors from the very right to the very left one by one we get a clear picture of how the ordered monomials with matching powers but different starting points turn into each other:

$$\begin{aligned} &\mathsf{w}_M^{p_M}\ldots\mathsf{w}_2^{p_2}\mathsf{w}_1^{p_1} \\ &= q^{2p_Mp_1}q^{-2p_1p_2}\ \mathsf{w}_1^{p_1}\mathsf{w}_M^{p_M}\ldots\mathsf{w}_3^{p_3}\mathsf{w}_2^{p_2} \\ &= q^{2p_Mp_1}q^{-2p_2p_3}\ \mathsf{w}_2^{p_2}\mathsf{w}_1^{p_1}\ldots\mathsf{w}_4^{p_4}\mathsf{w}_3^{p_4} = \ldots \end{aligned}$$

Now we know. The expression $q^{-2p_{m-1}p_m}\ \mathsf{w}_{m-1}^{p_{m-1}}\mathsf{w}_{m-2}^{p_{m-2}}\ldots\mathsf{w}_{m+1}^{p_{m+1}}\mathsf{w}_m^{p_m}$ does not depend on m provided $p_{\ell+M}\equiv p_\ell$. We award it with a self-explanatory notation

$$\overset{\circlearrowleft}{\prod}\mathsf{w}_\ell^{p_\ell} \equiv q^{-2p_Mp_1}\ \mathsf{w}_M^{p_M}\ldots\mathsf{w}_2^{p_2}\mathsf{w}_1^{p_1}$$

and extend this definition linearly to the corresponding polynomials, in particular,

$$\begin{aligned} &\overset{\circlearrowleft}{\prod}\epsilon(\mathsf{w}_\ell) \equiv \sum_{p_M,\cdots,p_2,p_1} c_{p_M}\cdots c_{p_2}c_{p_1}\overset{\circlearrowleft}{\prod}\mathsf{w}^{p_\ell} \\ &= \sum_{p_M,\cdots,p_2,p_1} c_{p_M}\cdots c_{p_2}c_{p_1}q^{-2p_Mp_1}\ \mathsf{w}_M^{p_M}\ldots\mathsf{w}_2^{p_2}\mathsf{w}_1^{p_1} \\ &= \sum_{p_M,p_1} c_{p_M}c_{p_1}q^{-2p_Mp_1}\ \mathsf{w}_M^{p_M}\Big(\epsilon(\mathsf{w}_{M-1})\ldots\epsilon(\mathsf{w}_3)\epsilon(\mathsf{w}_2)\Big)\mathsf{w}_1^{p_1} \end{aligned}$$

$$= \sum_{p_{m-1},p_m} c_{p_{m-1}} c_{p_m} q^{-2p_{m-1}p_m} \ \mathsf{w}_{m-1}^{p_{m-1}} \Big(\epsilon(\mathsf{w}_{m-2}) \ldots \epsilon(\mathsf{w}_{m+2}) \epsilon(\mathsf{v}_{m+1}) \Big) \mathsf{w}_m^{p_m}$$

with the coefficients c coming from

$$\epsilon(z) = \sum_p c_p z^p.$$

The same treatment applies as well to the 'selfdual' ϵ's of Section 5:

$$\boldsymbol{\epsilon}_q(z) = \sum_{p,r} c_{p,r} z^p z^{\theta r}$$

$$\prod^{\circlearrowleft} \boldsymbol{\epsilon}_q(\mathsf{w}_\ell) = \sum_{p_M,p_1,r_M,r_1} c_{p_M,r_M} c_{p_1,r_1} q^{-2(p_M p_1 + \theta^2 r_M r_1)}$$
$$\times \mathsf{w}_M^{p_M} \mathsf{w}_M^{\theta r_M} \Big(\boldsymbol{\epsilon}_q(\mathsf{w}_{M-1}) \ldots \boldsymbol{\epsilon}_q(\mathsf{w}_3) \boldsymbol{\epsilon}_q(\mathsf{w}_2) \Big) \mathsf{w}_1^{p_1} \mathsf{w}_1^{\theta r_1}.$$

So, we achieve the vital translational invariance of those products

$$\mathsf{S} \prod^{\circlearrowleft} \epsilon(\mathsf{w}_\ell) = \prod^{\circlearrowleft} \epsilon(\mathsf{w}_\ell) \ \mathsf{S}$$

sacrificing none of their 'orderness'. The time has come to plug the repaired evolution operator

$$\mathsf{Q} = \mathsf{SF} \prod^{\circlearrowleft} \epsilon(\mathsf{w}_\ell)$$

into the hypothetical equation ... see the beginning of this Section.

Now as we finally established that the operator Q is indeed responsible for the quantized and fully discretized Liouville equation

$$\mathsf{v}_{m+1,n+1} \mathsf{v}_{m,n} - q^{-1} \mathsf{v}_{m,n+1} \mathsf{v}_{m+1,n} = 1,$$

we must admit that so far the commitment to this particular equation was only a matter of personal taste. What would change if the function involved

$$\mathsf{Q} = \mathsf{SF} \prod^{\circlearrowleft} f(\mathsf{w}_\ell)$$

was different, for instance,

$$f(z) = \frac{\epsilon(z)}{\epsilon(q^{2\lambda} z)} \qquad ?$$

Nothing except the r.h.s. of the functional equation

$$\frac{f(qz)}{f(q^{-1}z)} = \frac{1+q^{2\lambda}z}{1+z}$$

and the form of the eventual equations of motion

$$\mathsf{v}_{m+1,n+1}\mathsf{v}_{m,n} - q^{-1}\mathsf{v}_{m,n+1}\mathsf{v}_{m+1,n} = 1 - q^{\lambda+1}\mathsf{v}_{m+1,n+1}\mathsf{v}_{m,n+1}\mathsf{v}_{m+1,n}\mathsf{v}_{m,n}.$$

By the way, this is another Hirota's equation, the discrete sine-Gordon one. We shall see it again in Section 11.

10 Classical continuum limit

The equation of Section 6 certainly turns into the 'light-cone' Liouville equation

$$\frac{\partial^2\psi}{\partial\xi\partial\tau} = e^{-2\psi},$$

the matching Cauchy problem being

$$\psi(\xi,\tau)|_{\tau=0} = \psi(\xi).$$

The algebra of observables from Section 7 transforms into no less familiar Poisson bracket reading

$$\{\psi(\xi),\psi(\eta)\} = \tfrac{1}{4}\mathrm{sign}(\xi-\eta).$$

The evolution operator of Section 8 bears some resemblance to the corresponding Hamiltonian

$$\mathcal{H} = \int d\xi\; e^{-2\psi}.$$

What is wrong? We seem to inherit also the quasiperiodic boundary condition

$$\psi(\xi+\pi) = \psi(\xi) + \Psi$$

which obviously contradicts to the equation. The periodic condition

$$\psi(\xi+\pi) = \psi(\xi)$$

could do but that in turn would contradict the Poisson bracket. A more careful examination reveals that the 'constant' in boundary conditions is not a constant of motion:

$$\mathsf{Q}^{-1}\mathsf{C}\mathsf{Q} = \mathsf{C}^{-1}.$$

Of course, the lattice equations of motion themselves have no problem with that. However, their solutions are not smooth enough to survive a straightforward continuum limit. Anyway, this peculiarity is not too relevant to what we are after in this paper.

11 Conservation laws

Looking at the two evolution operators we now possess

$$\mathfrak{Q} = \prod_{a=1}^{M} \epsilon(\mathfrak{w}_{2a-1}) \ \mathfrak{F} \prod_{a=1}^{M} \epsilon(\mathfrak{w}_{2a}) \qquad \mathsf{Q} = \mathsf{SF} \prod^{\circlearrowright} \epsilon(\mathsf{w}_\ell)$$

do we see something in the latter that was not there in the former? We do, the latter looks almost like a good old ordered product of 'fundamental R-matrices' (Tarasov *et al.*1983). According to (Volkov 1997a), the shift-n-flipless part of Q

$$\Omega = \prod^{\circlearrowright} \epsilon(\mathsf{w}_\ell)$$

belongs

$$\Omega = \Omega_{(\infty)}$$

in a family

$$\Omega_{(\lambda)} = \prod^{\circlearrowright} \epsilon(\lambda|\mathsf{w}_m) \qquad \epsilon(\lambda|z) \equiv \frac{\epsilon(z)}{\epsilon(q^{2\lambda}z)}$$

consolidated by the Artin-Yang-Baxter's Equation

$$\mathsf{R}_{m+1}(\lambda,\mu)\mathsf{R}_{m}(\lambda)\mathsf{R}_{m+1}(\mu) = \mathsf{R}_{m}(\mu)\mathsf{R}_{m+1}(\lambda)\mathsf{R}_{m}(\lambda,\mu).$$

The choice of notation

$$\mathsf{R}_{m}(\lambda) \equiv \epsilon(\lambda|\mathsf{w}_m) \qquad \mathsf{R}_{m}(\lambda,\mu) \equiv \frac{\mathsf{R}_{m}(\lambda)}{\mathsf{R}_{m}(\mu)}$$

is meant to emphasize the R-matrix connection. Let us recall how that AYBE could be verified. From (Faddeev and Volkov 1993) comes the multiplication rule

$$\epsilon(\lambda|\mathsf{b})\epsilon(\lambda|\mathsf{a}) = \epsilon(\lambda|\mathsf{a} + \mathsf{b} + q\mathsf{ba})$$

applying whenever a and b satisfy the Weyl's algebra

$$\mathsf{ab} = q^2\mathsf{ba}.$$

The two w's next to each other certainly do, therefore

$$\mathsf{R}_{m+1}(\lambda)\mathsf{R}_{m}(\lambda) = \epsilon(\lambda|\mathsf{w}_m + \mathsf{w}_{m+1} + q\mathsf{w}_{m+1}\mathsf{w}_m),$$

therefore

$$\mathsf{R}_{m+1}(\lambda)\mathsf{R}_{m}(\lambda)\mathsf{R}_{m+1}(\mu)\mathsf{R}_{m}(\mu)$$

$$= \epsilon(\lambda|\mathsf{w}_m + \mathsf{w}_{m+1} + q\mathsf{w}_{m+1}\mathsf{w}_m)\epsilon(\mu|\mathsf{w}_m + \mathsf{w}_{m+1} + q\mathsf{w}_{m+1}\mathsf{w}_m)$$

$$= \epsilon(\mu|\mathsf{w}_m + \mathsf{w}_{m+1} + q\mathsf{w}_{m+1}\mathsf{w}_m)\epsilon(\lambda|\mathsf{w}_m + \mathsf{w}_{m+1} + q\mathsf{w}_{m+1}\mathsf{w}_m)$$

$$= \mathsf{R}_{m+1}(\mu)\mathsf{R}_m(\mu)\mathsf{R}_{m+1}(\lambda)\mathsf{R}_m(\lambda).$$

This is it.

AYBE and ordered products are known to make a natural match

$$\frac{\mathsf{R}_{m+1}(\lambda)}{\mathsf{R}_{m+1}(\mu)}\Big(\mathsf{R}_m(\lambda)\mathsf{R}_{m-1}(\lambda)\dots\mathsf{R}_1(\lambda)\Big)\Big(\mathsf{R}_{m+1}(\mu)\mathsf{R}_m(\mu)\dots\mathsf{R}_2(\mu)\Big)$$

$$= \frac{\mathsf{R}_{m+1}(\lambda)}{\mathsf{R}_{m+1}(\mu)}\Big(\mathsf{R}_m(\lambda)\mathsf{R}_{m+1}(\mu)\Big)\Big(\mathsf{R}_{m-1}(\lambda)\mathsf{R}_m(\mu)\Big)\dots\Big(\mathsf{R}_1(\lambda)\mathsf{R}_2(\mu)\Big)$$

$$= \Big(\mathsf{R}_m(\mu)\mathsf{R}_{m+1}(\lambda)\Big)\frac{\mathsf{R}_m(\lambda)}{\mathsf{R}_m(\mu)}\Big(\mathsf{R}_{m-1}(\lambda)\mathsf{R}_m(\mu)\Big)\dots\Big(\mathsf{R}_1(\lambda)\mathsf{R}_2(\mu)\Big)$$

$$= \Big(\mathsf{R}_m(\mu)\mathsf{R}_{m+1}(\lambda)\Big)\Big(\mathsf{R}_{m-1}(\mu)\mathsf{R}_m(\lambda)\Big)\frac{\mathsf{R}_{m-1}(\lambda)}{\mathsf{R}_{m-1}(\mu)}\dots\Big(\mathsf{R}_1(\lambda)\mathsf{R}_2(\mu)\Big)$$

$$= \Big(\mathsf{R}_m(\mu)\mathsf{R}_{m+1}(\lambda)\Big)\Big(\mathsf{R}_{m-1}(\mu)\mathsf{R}_m(\lambda)\Big)\dots\frac{\mathsf{R}_2(\lambda)}{\mathsf{R}_2(\mu)}\Big(\mathsf{R}_1(\lambda)\mathsf{R}_2(\mu)\Big)$$

$$= \Big(\mathsf{R}_m(\mu)\mathsf{R}_{m+1}(\lambda)\Big)\Big(\mathsf{R}_{m-1}(\mu)\mathsf{R}_m(\lambda)\Big)\dots\Big(\mathsf{R}_1(\mu)\mathsf{R}_2(\lambda)\Big)\mathsf{R}_1(\lambda,\mu)$$

$$= \Big(\mathsf{R}_m(\mu)\mathsf{R}_{m-1}(\mu)\dots\mathsf{R}_1(\mu)\Big)\Big(\mathsf{R}_{m+1}(\lambda)\mathsf{R}_m(\lambda)\dots\mathsf{R}_2(\lambda)\Big)\frac{\mathsf{R}_1(\lambda)}{\mathsf{R}_1(\mu)},$$

so, one hopes the $\circlearrowleft$-ed product to step beyond $m = M - 2$ and deliver

$$\Omega(\lambda)\Omega(\mu) = \Omega(\mu)\Omega(\lambda).$$

Let us see.

$$\Omega(\lambda)\Omega(\mu)$$

$$= \sum c^{(\lambda)}_{p_M} c^{(\lambda)}_{p_1} q^{-2p_M p_1} \mathsf{w}_M^{p_M} \mathsf{R}^{(\lambda)}_{M-1} \dots \mathsf{R}^{(\lambda)}_2 \mathsf{w}_1^{p_1}$$

$$\times \sum c^{(\mu)}_{r_M} c^{(\mu)}_{r_1} q^{-2r_M r_1} \mathsf{w}_M^{r_M} \mathsf{R}^{(\mu)}_{M-1} \ldots \mathsf{R}^{(\mu)}_2 \mathsf{w}_1^{r_1}$$

$$= \sum c^{(\lambda)}_{p_M} c^{(\mu)}_{r_M} c^{(\lambda)}_{p_1} c^{(\mu)}_{r_1} q^{-2(p_M p_1 + r_M r_1 + r_M p_1)}$$

$$\times \mathsf{w}_M^{p_M} \mathsf{R}^{(\lambda)}_{M-1} \mathsf{w}_M^{r_M} \mathsf{R}^{(\lambda)}_{M-2} \mathsf{R}^{(\mu)}_{M-1} \ldots \mathsf{R}^{(\lambda)}_2 \mathsf{R}^{(\mu)}_3 \mathsf{w}_1^{p_1} \mathsf{R}^{(\mu)}_2 \mathsf{w}_1^{r_1}$$

- so far, we only recalled the definitions and did some reshuffling not involving any nontrivial commutation relations, (λ) and (μ) moved to the superscript level in order to save some paper -

$$= \sum c^{(\lambda)}_{p_M} c^{(\mu)}_{r_M} c^{(\lambda)}_{p_1} c^{(\mu)}_{r_1} q^{-2(p_M p_1 + r_M r_1 + r_M p_1)}$$

$$\times \mathsf{w}_M^{p_M} \mathsf{R}^{(\lambda)}_{M-1} \mathsf{w}_M^{r_M} \frac{\mathsf{R}^{(\mu)}_{M-1}}{\mathsf{R}^{(\lambda)}_{M-1}} \frac{\mathsf{R}^{(\lambda)}_{M-1}}{\mathsf{R}^{(\mu)}_{M-1}} \mathsf{R}^{(\lambda)}_{M-2} \mathsf{R}^{(\mu)}_{M-1} \ldots \mathsf{R}^{(\lambda)}_2 \mathsf{R}^{(\mu)}_3 \mathsf{w}_1^{p_1} \mathsf{R}^{(\mu)}_2 \mathsf{w}_1^{r_1}$$

- the unit operator $\frac{\mathsf{R}^{(\mu)}_{M-1}}{\mathsf{R}^{(\lambda)}_{M-1}} \frac{\mathsf{R}^{(\lambda)}_{M-1}}{\mathsf{R}^{(\mu)}_{M-1}} = \mathsf{1}$ has been inserted -

$$= \sum c^{(\lambda)}_{p_M} c^{(\mu)}_{r_M} c^{(\lambda)}_{p_1} c^{(\mu)}_{r_1} q^{-2(p_M p_1 + r_M r_1 + r_M p_1)}$$

$$\times \mathsf{w}_M^{p_M} \mathsf{R}^{(\lambda)}_{M-1} \mathsf{w}_M^{r_M} \frac{\mathsf{R}^{(\mu)}_{M-1}}{\mathsf{R}^{(\lambda)}_{M-1}} \mathsf{R}^{(\mu)}_{M-2} \mathsf{R}^{(\lambda)}_{M-1} \ldots \mathsf{R}^{(\mu)}_2 \mathsf{R}^{(\lambda)}_3 \frac{\mathsf{R}^{(\lambda)}_2}{\mathsf{R}^{(\mu)}_2} \mathsf{w}_1^{p_1} \mathsf{R}^{(\mu)}_2 \mathsf{w}_1^{r_1}$$

- AYBE did its habitual job -

$$= \sum c^{(\lambda)}_{p_M} c^{(\mu)}_{r_M} c^{(\lambda)}_{p_1} c^{(\mu)}_{r_1} q^{-2(p_M p_1 + r_M r_1 + r_M p_1)}$$

$$\times \mathsf{w}_M^{p_M} \mathsf{R}^{(\lambda)}_{M-1} \mathsf{w}_M^{r_M} \frac{\mathsf{R}^{(\mu)}_{M-1}}{\mathsf{R}^{(\lambda)}_{M-1}} \ldots \mathsf{R}^{(\mu)}_{m+1} \mathsf{R}^{(\lambda)}_{m+2} \mathsf{R}^{(\mu)}_{m} \mathsf{R}^{(\lambda)}_{m+1} \mathsf{R}^{(\mu)}_{m-1} \mathsf{R}^{(\lambda)}_{m} \ldots \frac{\mathsf{R}^{(\lambda)}_2}{\mathsf{R}^{(\mu)}_2} \mathsf{w}_1^{p_1} \mathsf{R}^{(\mu)}_2 \mathsf{w}_1^{r_1}$$

- nothing happened, we just refocused the attention to the middle portion of the product -

$$= \sum c^{(\lambda)}_{p_M} c^{(\mu)}_{r_M} c^{(\mu)}_{r_{m+1}} c^{(\mu)}_{r_m} c^{(\lambda)}_{p_{m+1}} c^{(\lambda)}_{p_m} c^{(\lambda)}_{p_1} c^{(\mu)}_{r_1} q^{-2(p_M p_1 + r_M r_1 + r_M p_1)}$$

$$\times \mathsf{w}_M^{p_M} \mathsf{R}^{(\lambda)}_{M-1} \mathsf{w}_M^{r_M} \frac{\mathsf{R}^{(\mu)}_{M-1}}{\mathsf{R}^{(\lambda)}_{M-1}} \dots \mathsf{w}_{m+1}^{r_{m+1}} \mathsf{R}^{(\lambda)}_{m+2} \mathsf{w}_m^{r_m} \mathsf{w}_{m+1}^{p_{m+1}} \mathsf{R}^{(\mu)}_{m-1} \mathsf{w}_m^{p_m} \dots \frac{\mathsf{R}^{(\lambda)}_2}{\mathsf{R}^{(\mu)}_2} \mathsf{w}_1^{p_1} \mathsf{R}^{(\mu)}_2 \mathsf{w}_1^{r_1}$$

- we disassembled some of the R's -

$$= \sum c^{(\lambda)}_{p_M} c^{(\mu)}_{r_M} c^{(\mu)}_{r_{m+1}} c^{(\lambda)}_{p_{m+1}} c^{(\mu)}_{r_m} c^{(\lambda)}_{p_m} c^{(\lambda)}_{p_1} c^{(\mu)}_{r_1} q^{-2(p_M p_1 + r_M r_1 + r_M p_1 - p_{m+1} r_m)}$$

$$\times \left(\mathsf{w}_M^{p_M} \mathsf{R}^{(\lambda)}_{M-1} \mathsf{w}_M^{r_M} \frac{\mathsf{R}^{(\mu)}_{M-1}}{\mathsf{R}^{(\lambda)}_{M-1}} \dots \mathsf{w}_{m+1}^{r_{m+1}} \mathsf{R}^{(\lambda)}_{m+2} \mathsf{w}_{m+1}^{p_{m+1}} \right)$$

$$\times \left(\mathsf{w}_m^{r_m} \mathsf{R}^{(\mu)}_{m-1} \mathsf{w}_m^{p_m} \dots \frac{\mathsf{R}^{(\lambda)}_2}{\mathsf{R}^{(\mu)}_2} \mathsf{w}_1^{p_1} \mathsf{R}^{(\mu)}_2 \mathsf{w}_1^{r_1} \right)$$

- the two in the middle traded places -

$$= \sum c^{(\mu)}_{r_m} c^{(\lambda)}_{p_m} c^{(\lambda)}_{p_1} c^{(\mu)}_{r_1} c^{(\lambda)}_{p_M} c^{(\mu)}_{r_M} c^{(\mu)}_{r_{m+1}} c^{(\lambda)}_{p_{m+1}} q^{-2(r_m r_{m+1} + p_m p_{m+1} + p_m r_{m+1} - r_1 p_M)}$$

$$\times \left(\mathsf{w}_m^{r_m} \mathsf{R}^{(\mu)}_{m-1} \mathsf{w}_m^{p_m} \dots \frac{\mathsf{R}^{(\lambda)}_2}{\mathsf{R}^{(\mu)}_2} \mathsf{w}_1^{p_1} \mathsf{R}^{(\mu)}_2 \mathsf{w}_1^{r_1} \right)$$

$$\times \left(\mathsf{w}_M^{p_M} \mathsf{R}^{(\lambda)}_{M-1} \mathsf{w}_M^{r_M} \frac{\mathsf{R}^{(\mu)}_{M-1}}{\mathsf{R}^{(\lambda)}_{M-1}} \dots \mathsf{w}_{m+1}^{r_{m+1}} \mathsf{R}^{(\lambda)}_{m+2} \mathsf{w}_{m+1}^{p_{m+1}} \right)$$

- the two halves in big brackets passed through each other -

$$= \sum c^{(\mu)}_{r_m} c^{(\lambda)}_{p_m} c^{(\lambda)}_{p_1} c^{(\lambda)}_{p_M} c^{(\mu)}_{r_1} c^{(\mu)}_{r_M} c^{(\mu)}_{r_{m+1}} c^{(\lambda)}_{p_{m+1}} q^{-2(r_m r_{m+1} + p_m p_{m+1} + p_m r_{m+1})}$$

$$\times \mathsf{w}_m^{r_m} \mathsf{R}^{(\mu)}_{m-1} \mathsf{w}_m^{p_m} \dots \frac{\mathsf{R}^{(\lambda)}_2}{\mathsf{R}^{(\mu)}_2} \mathsf{w}_1^{p_1} \mathsf{R}^{(\mu)}_2 \mathsf{w}_M^{p_M} \mathsf{w}_1^{r_1} \mathsf{R}^{(\lambda)}_{M-1} \mathsf{w}_M^{r_M} \frac{\mathsf{R}^{(\mu)}_{M-1}}{\mathsf{R}^{(\lambda)}_{M-1}} \dots \mathsf{w}_{m+1}^{r_{m+1}} \mathsf{R}^{(\lambda)}_{m+2} \mathsf{w}_{m+1}^{p_{m+1}}$$

- the two in the middle traded places -

$$= \sum c^{(\mu)}_{r_m} c^{(\lambda)}_{p_m} c^{(\mu)}_{r_{m+1}} c^{(\lambda)}_{p_{m+1}} q^{-2(r_m r_{m+1} + p_m p_{m+1} + p_m r_{m+1})}$$

$$\times \mathsf{w}^{r_m}_m \mathsf{R}^{(\mu)}_{m-1} \mathsf{w}^{p_m}_m \ldots \frac{\mathsf{R}^{(\lambda)}_2}{\mathsf{R}^{(\mu)}_2} \mathsf{R}^{(\lambda)}_1 \mathsf{R}^{(\mu)}_2 \mathsf{R}^{(\lambda)}_M \mathsf{R}^{(\mu)}_1 \mathsf{R}^{(\lambda)}_{M-1} \mathsf{R}^{(\mu)}_M \frac{\mathsf{R}^{(\mu)}_{M-1}}{\mathsf{R}^{(\lambda)}_{M-1}} \ldots \mathsf{w}^{r_{m+1}}_{m+1} \mathsf{R}^{(\lambda)}_{m+2} \mathsf{w}^{p_{m+1}}_{m+1}$$

- we assembled some R's, now it only remains to apply AYBE three more times -

$$= \sum c^{(\mu)}_{r_m} c^{(\lambda)}_{p_m} c^{(\mu)}_{r_{m+1}} c^{(\lambda)}_{p_{m+1}} q^{-2(r_m r_{m+1} + p_m p_{m+1} + p_m r_{m+1})}$$

$$\times \mathsf{w}^{r_m}_m \mathsf{R}^{(\mu)}_{m-1} \mathsf{w}^{p_m}_m \ldots \mathsf{R}^{(\mu)}_1 \mathsf{R}^{(\lambda)}_2 \mathsf{R}^{(\mu)}_M \mathsf{R}^{(\lambda)}_1 \mathsf{R}^{(\mu)}_{M-1} \mathsf{R}^{(\mu)}_M \ldots \mathsf{w}^{r_{m+1}}_{m+1} \mathsf{R}^{(\lambda)}_{m+2} \mathsf{w}^{p_{m+1}}_{m+1}$$

$$= \sum c^{(\mu)}_{r_m} c^{(\mu)}_{r_{m+1}} q^{-2 r_m r_{m+1}} \mathsf{w}^{r_m}_m \mathsf{R}^{(\mu)}_{m-1} \ldots \mathsf{R}^{(\mu)}_{m+2} \mathsf{w}^{r_{m+1}}_{m+1}$$

$$\times \sum c^{(\lambda)}_{p_m} c^{(\lambda)}_{p_{m+1}} q^{-2 p_m p_{m+1}} \mathsf{w}^{p_m}_m \mathsf{R}^{(\lambda)}_{m-1} \ldots \mathsf{R}^{(\lambda)}_{m+2} \mathsf{w}^{p_{m+1}}_{m+1}$$

$$= \Omega(\mu)\Omega(\lambda).$$

Done, at least for $M > 5$. In fact, even $M = 3$ is possible but this would take three more pages to verify. Anyway, a more civilized edition of the above proof is presented in (Volkov 1997b).

The commutativity of the Ω's may be a good news but there is a bad one too. The flip operator F does not commute with the Ω's. Which means there is another family

$$\mho(\lambda) = \mathsf{F}^{-1}\Omega(\lambda)\mathsf{F} = \mathsf{F}\Omega(\lambda)\mathsf{F}^{-1}$$

not coinciding with the original one. Of course, the $\mho$'s commute with each other

$$\mho(\lambda)\mho(\mu) = \mho(\mu)\mho(\lambda)$$

but it is not immediately clear whether

$$\Omega(\lambda)\mho(\mu) = \mho(\mu)\Omega(\lambda)$$

should also be true. Fortunately, there is some hidden agenda making it happen. Technically-wise, the proof is the same as that above except the AYBE in use is somewhat different:

$$\left(\mathsf{w}^{\mu}_{m+1}\epsilon(\lambda - \mu | \mathsf{w}_{m+1})\right)\epsilon(\lambda|\mathsf{w}_m)\epsilon(\mu|\mathsf{w}^{-1}_{m+1})$$

$$= \epsilon(\lambda-\mu|\mathsf{w}_{m+1})\epsilon(\lambda|q^{-2\mu}\mathsf{w}_m)\mathsf{w}_{m+1}^{\mu}\epsilon(\mu|\mathsf{w}_{m+1}^{-1})$$

$$= \frac{\epsilon(\lambda|q^{-2\mu}\mathsf{w}_{m+1})}{\epsilon(\mu|q^{-2\mu}\mathsf{w}_{m+1})}\epsilon(\lambda|q^{-2\mu}\mathsf{w}_m)q^{\mu^2}\epsilon(\mu|q^{-2\mu}\mathsf{w}_{m+1})$$

$$= q^{\mu^2}\epsilon(\mu|q^{-2\mu}\mathsf{w}_m)\epsilon(\lambda|q^{-2\mu}\mathsf{w}_{m+1})\frac{\epsilon(\lambda|q^{-2\mu}\mathsf{w}_m)}{\epsilon(\mu|q^{-2\mu}\mathsf{w}_m)}$$

$$= \mathsf{w}_m^{\mu}\epsilon(\mu|\mathsf{w}_m^{-1})\epsilon(\lambda|q^{-2\mu}\mathsf{w}_{m+1})\epsilon(\lambda-\mu|\mathsf{w}_m)$$

$$= \epsilon(\mu|\mathsf{w}_m^{-1})\epsilon(\lambda|\mathsf{w}_{m+1})\Big(\epsilon(\lambda-\mu|\mathsf{w}_m)\mathsf{w}_m^{\mu}\Big).$$

Once the flip-n-shift join

$$\mathsf{Q}(\lambda) = \mathsf{SF}\Omega(\lambda) = \mho(\lambda)\mathsf{SF}$$

one realizes that full commutativity is not there

$$\mathsf{Q}(\lambda)\mathsf{Q}(\mu) \neq \mathsf{Q}(\mu)\mathsf{Q}(\lambda),$$

only ‘squares’ can do:

$$\Big(\mathsf{Q}(\kappa)\mathsf{Q}(\lambda)\Big)\Big(\mathsf{Q}(\mu)\mathsf{Q}(\nu)\Big) = \mathsf{F}\Omega(\kappa)\mathsf{F}\Omega(\lambda)\mathsf{F}\Omega(\mu)\mathsf{F}\Omega(\nu)$$

$$= \mathsf{F}^4\mho(\kappa)\Omega(\lambda)\mho(\mu)\Omega(\nu) = \mathsf{F}^4\mho(\mu)\Omega(\nu)\mho(\kappa)\Omega(\lambda)$$

$$= \mathsf{F}\Omega(\mu)\mathsf{F}\Omega(\nu)\mathsf{F}\Omega(\kappa)\mathsf{F}\Omega(\lambda) = \Big(\mathsf{Q}(\mu)\mathsf{Q}(\nu)\Big)\Big(\mathsf{Q}(\kappa)\mathsf{Q}(\lambda)\Big).$$

In particular,

$$\mathsf{Q}^{-2}\mathsf{Q}^2(\lambda)\mathsf{Q}^2 = \mathsf{Q}^2(\lambda).$$

On these grounds, let us call the $\mathsf{Q}(\lambda)$'s ‘conservation laws’ even though what actually happens is that only their squares are only recovered on every other step in time. This peculiarity has everything to do with that discussed in Section 9. Apparently, not one but two time steps should make a ‘physical’ time unit.

12 Conclusion

We developed here the scheme allowing to describe some quantum dynamical systems in discrete 1+1-dimensional space-time. The discretized Liouville model was taken as the main example and treated both for laboratory coordinates and light-like ones. All considerations were purely algebraic, no representation and/or Hilbert space was used. We confined ourselves to pure Heisenberg picture of quantum theory.

The main outcome is the construction of evolution operator realizing the automorphism of the algebra of observables leading to the Heisenberg equations of motion representing lattice and quantum deformation of the corresponding classical equations. In this construction the famous q-exponent (q-dilogarithm) played the most prominent part.

We discussed also the integrability of the model presenting the set of conservation laws. Their construction and the verification of commutativity was based on a new solution of Artin-Yang-Baxter relation, being a close relative of the q-exponent.

We hope that the scheme of this paper is general enough and allows to include many related models of quantum theory. Our papers mentioned in Introduction give some illustration of this.

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