ALGEBRAIC ASPECTS OF BETHE-ANSATZ

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Abstract

In these lectures the introduction to algebraic aspects of Bethe Ansatz is given. The applications to the seminal spin 1/2 XXX model is discussed in detail and the generalization to higher spin as well as XXZ and lattice Sine-Gordon model are indicated. The origin of quantum groups and their appearance in CFT models is explained. The text can be considered as a guide to the research papers in this field.

Preface

During last few years I have given several short lecture courses on the application of (algebraic) Bethe Ansatz to the integrable models of quantum field theory in $1+1$ dimensional space-time, see [1], [2], [3]. Being essentially similar in spirit they differ in the choice of particular topics and the technical details. This course, which was delivered at the Institute of Theoretical Physics, University of New York at Stony Brook, is self-contained, and do not copy [1]-[3]. It can be considered as a short and non-technical introduction into the subject of integrable models. Interested students must turn to the vast literature. A minimal list of references is appended. A lot of references to historic as well as research papers could be found therein.

The job of taking notes of the course and producing the lecture notes was undertaken by Kostas Skenderis. Without his help this text would not appear. I am very grateful to him for his devoted work.

I would like to acknowledge the hospitality which was provided to me by Professor C.N. Yang in the Institute for Theoretical Physics, University of New York at Stony Brook.

1 Lecture 1

1.1 Introduction.

There are many different historical directions which led to the subject of exactly solvable model: (1) the study of magnetic chains initiated by Bethe in the early thirties, (2) the work of Onsanger and Baxter in 2 dimensional classical statistical mechanics, (3) scattering theory in many body problem with factorizable S-matrix (Berezin, Yang). Here we will follow one more path which again leads to exactly solvable models, namely the inverse scattering method.

Solitons are discovered almost a hundred years ago. These are particle-like solutions of non-linear differential equations. The prototype equation is the KdV (Korteweg-de Vries) equation

$$u_t = uu_x + u_{xxx}, \tag{1}$$

where u_t and u_x denote differentiation with respect to t and x, respectively. It has solution which describes a localized wave which moves without dissipation. The KdV equation resurrected in the early sixties through the work of Kruskal and others. They are the ones who named the particle-like solution soliton.

What is very interesting is the existence of many-soliton solutions which opens new possibilities for the particle spectrum. In contrast with the usual field theory where to each field corresponds just one particle, here we have a novel situation where one field might generate more than one particle. This possibility might be familiar nowdays after the emergence of the string theory but back in the early seventies the soliton mechanism of emergence of mass spectrum was the first example of circumvention of the perturbative paradigm.

Using the inverse scattering method, many different theories in $1 + 1$ dimensions were studied and solved, see for example monograph [4]. Among them the Bose Gas model, whose equation of motion is a nonlinear Schrödinger (NS) equation

$$i\frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial x^2} + g|\Psi|^2\Psi, \tag{2}$$

where $\Psi(x, t)$ is a complex field and g is the coupling constant, and the Sine-Gordon model. The latter describes a relativistic real field $\phi(x, t)$ whose

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equation of motion is given by

$$\Box\phi + \frac{m^2}{\beta}\sin\beta\phi = 0. \tag{3}$$

The constants m^2 and β are a mass parameter and the coupling constant of the model, respectively. The solution of this model revealed a very interesting spectrum with the mass of the particles given by the formula

$$m_n = \frac{16m}{\gamma}\sin\frac{n\gamma}{16}, \tag{4}$$

where $\gamma = 8/\beta^2$. These studies finally led to the so-called algebraic Bethe ansatz (1978).

A natural question is what a $1+1$ dimensional system has to do with the real world which is 4 dimensional. First of all, string theory teach us that 2D physics (worldsheet) can be ultimately related with higher dimensional world (target space). In fact, string theories are described by conformal field theories (CFT) which can be viewed as ultraviolet limit of exactly solvable models. (Alternatively, one might think the massive exactly solvable models as deformations of CFT's).

A second example where $1+1$ physics enters comes from 4D QCD. There are evidences that the high energy scattering is mainly 2D phenomenon (Lipatov, Verlinde-Verlinde).

The similarities between non-linear sigma model in 2D and Yang-Mills theories in 4D (asymptotic freedom, mass through dimensional transmutation etc.) gives one more motivation for studying 2D models. One hopes that 2D physics can teaches us how to tackle corresponding 4D problems.

1.2 The XXX model.

Consider the one-dimensional quantum periodic chain with N sites. We associate with each site n a local spin variable

$$\vec{s} = \frac{1}{2}\vec{\sigma}, \tag{5}$$

where the components of the vector $\vec{\sigma}$ are the Pauli matrices. The spin variables act on the Hilbert space $h = \mathbf{C}^2$. The full Hilbert space is the

tensor product of all local Hilbert spaces

$$\mathcal{H} = h^{\otimes N}. \tag{6}$$

Its dimension is $\dim(\mathcal{H}) = 2^N$. The spin variable act on the full Hilbert space $\mathcal{H}$ as follows

$$\vec{s}_n = I \otimes I \cdots \otimes \vec{s} \otimes \cdots I, \tag{7}$$

where I is the unit operator in the local Hilbert space h. The periodicity of the lattice means that $\vec{s}_{n+N} = \vec{s}_n$. The Hamiltonian of the system is given by

$$H = \sum_{n=1}^{N} \left[(s_n, s_{n+1}) - \frac{1}{4} \right], \tag{8}$$

where (s_n, s_{n+1}) is the inner product between $\vec{s}_n$ and $\vec{s}_{n+1}$ defined in the space of two spins, namely $\mathrm{C}^2 \otimes \mathrm{C}^2$. Notice that the Hamiltonian is manifestly negative.

Notice that we have done all possible regularizations. The lattice spacing Δ and the finite volume N provides natural ultraviolet and infrared cut-offs in the theory whereas the choice of the quantum spin as our variable renders the Hilbert space finite dimensional. After we solve the theory we will wish to take the thermodynamic limit $N \to \infty$. This is a non-trivial limit and it is related with the theory of infinite tensor product. However, in some cases of physical interest things are simpler. For example, in the case of ferromagnet the configuration consists mostly of, say, spin up configurations so only a finite number of spins could be turned down.

We shall use the permutation operator

$$P = \frac{1}{2}(I \otimes I + \vec{\sigma} \otimes \vec{\sigma}). \tag{9}$$

If we choose as a basis in C^2 the vectors

$$|+ >= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |- >= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10}$$

then the permutation operator is given by the matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

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in the basis $|++>, |+->, |-+>, |-->$. The Hamiltonian can be written in terms of the permutation operator as

$$H = \frac{1}{2}\sum_{n=1}^{N}(P_{n,n+1} - 1). \tag{12}$$

We now introduce the Lax operators $L_{n,a}$. Their significance will become clear later on. They are operator which act on the local space $h_n \otimes V$ (now we use the notation h_n for a local quantum space assigned to the site n), where V is an auxiliary space. In our example $h = \mathbf{C}^2$ is the same as V but in general it does not have to coincide with it. The $L_{n,a}$ is given by

$$
\begin{aligned}
L_{n,a}(\lambda) &= \lambda(I_n \otimes I_a) + i(\vec{s}_n \otimes \vec{\sigma}_a) \tag{13}\\
&= (\lambda - \frac{i}{2})I + iP_{n,a} \tag{14}\\
&= \begin{pmatrix} \lambda + is_n^3 & is_n^- \\ is_n^+ & \lambda - is_n^3 \end{pmatrix}, \tag{15}
\end{aligned}
$$

where $s^{\pm} = s^1 \pm is^2$ and λ is complex parameter. The index n refers to the site position whereas the index a is always an auxiliary space index. The role of the parameter λ will be made clear later. In the last line we treat $L_{n,a}$ as an matrix on V with coefficients as operators in h_n.

The Lax operator can be interpreted as a connection on the 1D lattice. Let ϕ_n be vectors from $\mathcal{H} \otimes V$ assigned to each lattice site. We say that ϕ_n is parallel if

$$\phi_{n+1} = \frac{1}{\lambda}L_n\phi_n, \tag{16}$$

(where the factor $1/\lambda$ is a suitable rescaling), so that the Lax operator just realizes parallel transport. If we take the formal classical continuous limit $\Delta \to 0$, $\hbar \to 0$ (of course after we reinstate the $\hbar$ dependence) and define the continuous spin variable as

$$s(x) = \lim_{\Delta \to 0} \frac{s_n}{\Delta}; \ x = n\Delta, \tag{17}$$

then using (13) and (16) we get

$$\frac{1}{i}\phi'(x) = \lim_{\Delta \to 0}\frac{1}{i}\frac{\phi_{n+1} - \phi_n}{\Delta} = \frac{S(x)}{\lambda}\phi(x) \tag{18}$$

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where $S(x) = (\vec{s}(x), \vec{\sigma})$. But this is the equation which was used in the inverse scattering method applied to magnet by Takhajan[4]. So (16) is a natural lattice and quantum deformation of (18).

The next task is to establish commutation relations between the Lax operators. First we introduce our notation

$$A_{ik}B_{mn} \equiv (A \otimes B)_{im|kn} \equiv (A^1 B^2)_{im,kn},$$

where $A^1 = A \otimes I$ and $B^2 = I \otimes B$. Since $L_{n,a}$ is a 4×4 matrix there are 16 matrix elements whose commutation relations we want to study. All these commutation relations can be compactly written as

$$R_{a_1,a_2}(\lambda - \mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu). \qquad (19)$$

This is an equation in $V_1 \otimes V_2 \otimes h_n$. The indices a_1 and a_2 and the variables λ and μ are associated with the auxiliary spaces V_1 and V_2, respectively. The matrix R_{a_1,a_2} governs the commutation relation of the matrix elements of Lax operators. In our case it is given by

$$R_{a_1,a_2}(\lambda) = \frac{1}{\lambda + i}(\lambda I_{a_1,a_2} + iP_{a_1,a_2}). \qquad (20)$$

There is a nice graphical representation of (19). The Lax operator can be represented as a cross of two lines (fig. 1), one which represents the auxiliary space index (thin line) and one which represent the site index (thick line). Then the graphical representation of (19) is given in figure 2. The right hand of (19) is obtained from the left hand side by just shifting the n-line to the other side of the cross point of the remaining lines.
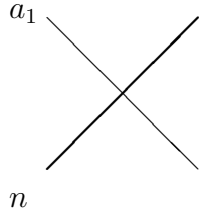

Figure 1. Graphical representation of the Lax operator L_{n,a_1}.
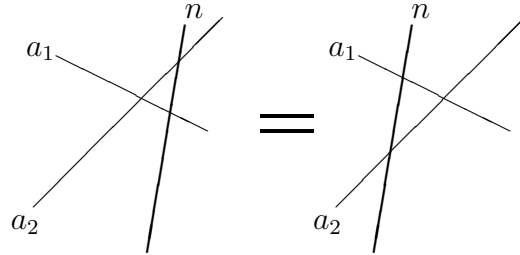
Figure 2. Graphical representation of Fundamental Commutation Relations (19).

Having interpreted the Lax operator as a kind of connection along the chain, we can introduce an operator which describes parallel transport once around the chain. This operator is the monodromy matrix $T_a(\lambda)$

$$T_a(\lambda) = L_{N,a}(\lambda)L_{N-1,a}(\lambda)\cdots L_{1,a}(\lambda). \tag{21}$$

The monodromy matrix satisfies commutation relations identical with the ones satisfied by the Lax operators, namely

$$R_{a_1,a_2}(\lambda - \mu)T_{a_1}(\lambda)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu). \tag{22}$$

These commutation relations can be easily proven by repeatedly use of (19). The graphical representation of (22) (fig. 3) gives an alternative (and easier) proof of (22) (the "train" argument).
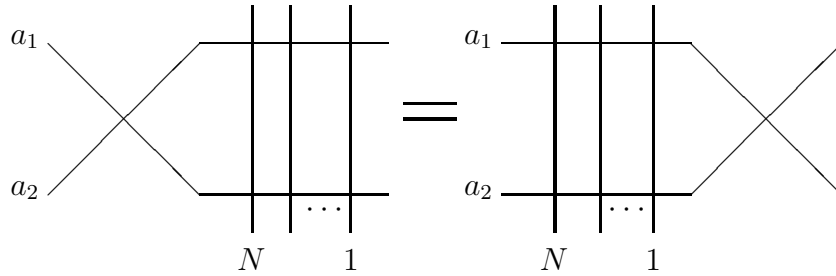

Figure 3. The "train" argument.

We introduce a family of operators

$$F(\lambda) = \text{Tr}_a(T_a(\lambda)), \tag{23}$$

where the trace is over the auxiliary space. The operator $F(\lambda)$ is acts on the full Hilbert space $\mathcal{H}$ (global operator). From its definition it follows that $F(\lambda)$ is a polynomial in λ of degree N. It follows from (22) that $F(\lambda)$ and $F(\mu)$ commute

$$[F(\lambda), F(\mu)] = 0, \tag{24}$$

so there are N independent quantities which mutually commute. As we will see in a moment the Hamiltonian is among them. So we have N integrals of

motion in involution for a system with N degrees of freedom. In the classical case it corresponds to the Liouville's definition of integrability.

Consider the Lax operator at the specific point $\lambda = i/2$. From (14) we get

$$L_{n,a}(\lambda = i/2) = iP_{n,a}. \tag{25}$$

Hence,

$$F(\lambda = i/2) = i^N \text{Tr}_a(P_{N,a}P_{N-1,a} \cdots P_{1,a}). \tag{26}$$

Using $P_{n,a}P_{m,a} = P_{m,n}P_{n,a}$, $P_{n,m} = P_{m,n}$ and $\text{Tr}_a P_a = I$ we get

$$F(\lambda = i/2) = i^N P_{1,2}P_{2,3} \cdots P_{N,N-1}. \tag{27}$$

But the right hand side is just the shift operator

$$U = P_{1,2}P_{2,3} \cdots P_{N,N-1}, \tag{28}$$

which has the property that,

$$UX_n = X_{n+1}U, \tag{29}$$

for every local operator X_n. It follows that U is just the exponential of the momentum operator Π

$$U = \exp i\Pi. \tag{30}$$

We now want to express the Hamiltonian in terms of the operators $F(\lambda)$. Since

$$\frac{d}{d\lambda}L_{n,a} = I_{n,a},$$

we get

$$\left.\frac{dF}{d\lambda}\right|_{\lambda=i/2} = i^{N-1} \sum_{j=1}^{N} \text{Tr}_a(P_{N,a} \cdots \hat{P}_{j,a} \cdots P_{1,a}), \tag{31}$$

where the hat denotes omission. Comparing (27) and (12) we see that the Hamiltonian can be written as

$$H = \left.\frac{i}{2}\frac{dF}{d\lambda}F^{-1}(\lambda)\right|_{\lambda=i/2} - \frac{N}{2}. \tag{32}$$

Notice that since the $F(\lambda)$ commute for different λ there is no problem with definition of the logarithmic derivative.

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Let the monodromy matrix be

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{33}$$

We want to diagonalize

$$F(\lambda) = A(\lambda) + D(\lambda). \tag{34}$$

It is easy to show that

$$\Omega = \prod_{n=1}^{N} \bigotimes |+>_n . \tag{35}$$

is an eigenvector of $F(\lambda)$. Indeed, since

$$s_n^+|+>= 0, \tag{36}$$

we get that $C(\lambda)\Omega = 0$, and consequently that the matrix $T(\lambda)\Omega$ is a triangular matrix. Furthermore, using (15), we get

$$A(\lambda)\Omega = (\lambda + \frac{i}{2})^N \Omega, \tag{37}$$

$$D(\lambda)\Omega = (\lambda - \frac{i}{2})^N \Omega \tag{38}$$

so that

$$F(\lambda)\Omega = [(\lambda + \frac{i}{2})^N + (\lambda - \frac{i}{2})^N]\Omega. \tag{39}$$

Now we are going to show that the $B(\lambda)$ can be used as spectrum raising operator. This will be done in the next lecture.

2 Lecture 2

In the last lecture we have seen that the vector Ω which consist of spin up configuration in all lattice sites is an eigenvector of $F(\lambda)$. The vector Ω plays a role similar with the vacuum state of the harmonic oscillator. To generate other states we act with $B(\lambda)$ on Ω. In that sense the operator $B(\lambda)$ is a kind of raising operator. However, the new vectors will be eigenvector of $F(\lambda)$ only for specific values of λ's.

Consider the state

$$\Phi(\{\lambda\}) = B(\lambda_1)B(\lambda_2)\cdots B(\lambda_l)\Omega. \tag{40}$$

To check if this it is an eigenvector of $A(\lambda)$ and $D(\lambda)$ we need the commutation relation between $A(\lambda)$, $D(\lambda)$ and $B(\lambda)$. These are obtained from (21) and are given by

$$[B(\lambda), B(\mu)] = 0, \tag{41}$$

$$A(\lambda)B(\mu) = \alpha(\lambda - \mu)B(\mu)A(\lambda) + \beta(\lambda - \mu)B(\lambda)A(\mu), \tag{42}$$

$$D(\lambda)B(\mu) = \gamma(\lambda - \mu)B(\mu)D(\lambda) + \delta(\lambda - \mu)B(\lambda)D(\mu), \tag{43}$$

where $\alpha(\lambda) = (\lambda - i)/\lambda$ and $\gamma(\lambda) = (\lambda + i)/\lambda$. The specific values of β and δ are irrelevant for our discussion. From (41) we see that the Φ is symmetric function of λ's. If the second term in the right hand side of (42) and (43) was absent then the vector $\Phi(\{\lambda\})$ would have been an eigenvector of $A(\lambda)$ and $D(\lambda)$ for all values of λ's. However, the presence of these terms generates extra unwanted terms when we commute the $A(\lambda)$ (or $D(\lambda)$) to the right. For a particular set of λ's all extra terms vanish and $\Phi(\{\lambda\})$ is, at the end, an eigenvector of $F(\lambda)$. This set is determined by the solution of a transcendental equation. For this particular set of λ's the eigenvalue equation reads

$$F(\lambda)\Phi(\{\lambda\}) = \Big[\prod_{m=1}^{l} \Big(\frac{\lambda_m - \lambda + i}{\lambda_m - \lambda}\Big)(\lambda + \frac{i}{2})^N + \prod_{m=1}^{N} \Big(\frac{\lambda_m - \lambda - i}{\lambda_m - \lambda}\Big)(\lambda - \frac{i}{2})^N \Big]\Phi(\{\lambda\}). \tag{44}$$

Since the left hand side of (44) is an analytic function of λ, so has to be the right hand side. However, the right hand side has poles at $\lambda = \lambda_l$. All the poles are removed if the λ's obey the equations

$$(\frac{\lambda_k + i/2}{\lambda_k - i/2})^N = \prod_{m=1; m\neq k}^{l} \frac{\lambda_k - \lambda_m + i}{\lambda_k - \lambda_m - i} \qquad k = 1, \ldots, l. \tag{45}$$

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In fact, this is exactly the condition for the cancellation of the unwanted extra terms mentioned above (one can find a formal proof in [5]. Thus, self-consistency of (44) automatically provides the set of λ's for which it is valid.

Consider the shift operator

$$U = e^{i\Pi} = i^{-N} F(\lambda)\Big|_{\lambda=i/2}. \tag{46}$$

We apply U to Φ. Since the second term in right hand side of (44) vanishes at $\lambda = i/2$, the spectrum of U is multiplicative and, hence, the spectrum of the momentum operator Π is additive

$$\Pi\Phi = \sum_{m=1}^{l} p(\lambda_m)\Phi, \tag{47}$$

where

$$p(\lambda_m) = \frac{1}{i} \ln \frac{\lambda_m + i/2}{\lambda_m - i/2}. \tag{48}$$

For λ real the momentum ranges over $[0, 2\pi]$. The spectrum of the Hamiltonian is also additive

$$H\Phi = \sum_{m=1}^{l} h(\lambda_m)\Phi, \tag{49}$$

where

$$h(\lambda) = \frac{1}{2}\frac{dp(\lambda)}{d\lambda} = -\frac{1}{2}\frac{1}{\lambda^2 + 1/4} \leq 0. \tag{50}$$

The parameter λ can be interpreted as the "rapidity". (Recall that the energy and the momentum of a relativistic particle can be parametrized in terms of the rapidity θ as $p = m\sinh\theta$, $E = m\cosh\theta$.) Eliminating the parameter λ we get the dispersion relation

$$h(p) = \frac{1}{2}(\cos p - 1). \tag{51}$$

The momentum of the system is quantized due to the fact that we have our system in the finite box. This becomes manifest if we rewrite (45) as

$$e^{ip(\lambda_k)N} = \prod_{m=1}^{l} S(\lambda_k - \lambda_m), \tag{52}$$

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where

$$S(\lambda) = \frac{\lambda + i}{\lambda - i},\tag{53}$$

is the two-particle scattering amplitude. (Compare with the free particle case where the quantization equation is $\exp ipN = 1$.) The fact that the l-particle scattering amplitude in the right hand side of (52) is expressed in terms of two-particle one is a manifestation of the integrability of the model.

Since the Hamiltonian is negative, the particle spectrum does not describe physical particles. However, if we reverse the sign of the Hamiltonian (ferromagnetic case) then we get physical particles which are called magnons.

Another observable of our system is the global spin. The spin operator is defined as

$$\vec{\Sigma} = \sum_{n=1}^{N} \vec{s}_n.\tag{54}$$

The spin operator appears in the $1/\lambda$ expansion of the monodromy matrix

$$T(\lambda) = \lambda^N \Big[I + \frac{\vec{\Sigma} \cdot \vec{\sigma}_a}{\lambda} + \mathcal{O}(\frac{1}{\lambda^2}) \Big].\tag{55}$$

The $\mathcal{O}(1/\lambda^2)$ term is related with Yangian symmetries[6]. Consider now the commutation relations (21) in the limit $\lambda \to \infty$, μ fixed. It follows that

$$[\frac{1}{2}\vec{\sigma}_a + \vec{\Sigma}, T_a(\mu)] = 0,\tag{56}$$

which means that the monodromy matrix is invariant under combined rotations in the full quantum and auxiliary space. One have in particular the following relations

$$[\vec{\Sigma}, F(\lambda)] = 0,\tag{57}$$

which implies rotational invariance of the Hamiltonian H,

$$[\Sigma_3, B(\lambda)] = -B(\lambda),\tag{58}$$

which means that the operator $B(\lambda)$ turns down one spin and

$$[\Sigma_+, B(\lambda)] = A(\lambda) - D(\lambda).\tag{59}$$

We evidently have

$$\Sigma_3 \Omega = \frac{N}{2}\Omega,\tag{60}$$

so that (58) leads to

$$\Sigma_3 \Phi = (\frac{N}{2} - l)\Phi, \tag{61}$$

where l is the number of B-operators acting on Ω (see (40)). There are many comments in order here. First of all notice the difference between the odd and even case. When N is even we get integer spin states, whereas when it is odd we get semi-integer spin states. In particular the ground state can only appear in the even case. So contrary to the usual intuition, the nature of spectrum is different for the N even and N odd case even in the limit $N \to \infty$. Next we can prove using (59) and the Bethe ansatz equations (45) that all Φ states are highest weight states, i.e.

$$\Sigma_+ \phi = 0. \tag{62}$$

To get the rest of the states we act with Σ_-. Since Σ_- commutes with the Hamiltonian H all the descendants have the same energy. Furthermore, equation (61) and (62) imply that $l \leq N/2$.

One can easily check that (44) is invariant under complex conjugation. This means that for every complex solution λ of (44), $\bar{\lambda}$ is also a solution. In addition all λ's are different (Pauli principle)[7]. This can be inferred by examining again the self-consistency of (45). If two λ's were equal then (45) would have a double pole. So we would need $(l+1)$ equations in order to remove all poles which is an over-determined system (we have only l unknowns). In fact one can prove that in the large N limit the $(l+1)$ equations are incompatible. So the λ's ought to be different.

We want to study the vacuum state. From (61) we infer that $l = N/2$. In addition, more detailed investigation shows that all λ's are real. We take the logarithm in both sides in (45) and we use a branch of the logarithm in the form

$$\frac{1}{i} \ln \left(\frac{\lambda + ia}{\lambda - ia}\right) = \pi - \arctan \frac{\lambda}{a}. \tag{63}$$

Equation (45) becomes

$$N \arctan 2\lambda_j = \sum_{l=1}^{N/2} \arctan(\lambda_j - \lambda_l) + \pi Q_j, \tag{64}$$

where Q_j are quantum numbers which parametrize the λ's. They are integers or half-integers. One can prove that λ is a monotonic function of Q_j so that

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$Q_j \to Q_j^{\max}$ in the limit $\lambda \to \infty$. In fact, for finite N the range of Q is

$$-\frac{N}{4} + \frac{1}{2} \le Q \le \frac{N}{4} - \frac{1}{2}, \tag{65}$$

so there $N/2$ Q's. This means that Q takes all the values from $-Q_{\max}$ to $Q_{\max}$, so actually we have $Q_j = j$. The ground state consists of a completely filled Dirac sea.

We now wish to take the thermodynamic limit $N \to \infty$. We introduce a new variable $x = j/N$. In the thermodynamic limit $\lambda_j \to \lambda(x)$. Assuming regularity of the function $\lambda(x)$ one can replace the sum in (64) by an integral within an $\mathcal{O}(1/N^2)$ error:

$$\arctan 2\lambda(x) = \pi x + \int_{-1/4}^{1/4} dy \arctan \left[\lambda(x) - \lambda(y)\right]. \tag{66}$$

This a non-linear equation but it can be reduced to a linear one as we will now show. Differentiating equation (66) w.r.t. x we get

$$\frac{2\lambda'(x)}{1 + 4\lambda^2} = \pi + \lambda'(x) \int \frac{1}{1 + (\lambda(x) - \lambda(y))^2} dy. \tag{67}$$

We introduce the density of states, $\rho(\lambda)$, defined as

$$\rho(\lambda) = \frac{1}{\lambda'(x)}. \tag{68}$$

This is a natural definition since $dx = \rho(\lambda)d\lambda$. Using (68), equation (67) becomes a linear integral equation for the density of states $\rho(\lambda)$:

$$\frac{2}{1 + 4\lambda^2} = \pi \rho(\lambda) + \int_{-\infty}^{\infty} \frac{1}{1 + (\lambda - \mu)^2} \rho(\mu) d\mu. \tag{69}$$

This equation was obtained by L. Hulthen can be solved by means of Fourier transform.

The ground state energy in the thermodynamic limit is given by

$$\begin{aligned} E &= \sum h(\lambda_l) \to N \int h(\lambda(x)) dx \\ &= N \int_{-\infty}^{\infty} h(\lambda)\rho(\lambda)d\lambda. \end{aligned} \tag{70}$$

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It can be shown that it is equal to

$$E_0 = -N \ln 2 = -\ln 2^N. \tag{71}$$

This value is related with the residual entropy at $T = 0$.

Let us now consider excitations. In the N-even case the first excitation appears when $l = N/2 - 1$ (triplet state) and it consists of two vacancies(holes) in the Fermi sea. Each of them carries spin $1/2$. The parameters Q are modified

$$Q_j = j + \theta(j - j_1) + \theta(j - j_2), \tag{72}$$

where θ is the Heaviside function. The parameters Q_j have two jumps at the position of the holes. Equation (66) is modified accordingly

$$\arctan 2\lambda(x) = \pi x + \frac{\pi}{N}[\theta(j - j_1) + \theta(j - j_2)] + \int_{-1/4}^{1/4} dy \arctan [\lambda(x) - \lambda(y)]. \tag{73}$$

and, hence,

$$\frac{2}{1 + 4\lambda^2} = \pi\rho(\lambda) + \frac{\pi}{N}[\delta(\lambda - \lambda_1) + \delta(\lambda - \lambda_2)] + \int_{-\infty}^{\infty} \frac{1}{1 + (\lambda - \mu)^2}\rho(\mu)d\mu, \tag{74}$$

where λ_i, $i = 1, 2$, are the rapidities for the holes. Thus, the density of states is also modified

$$\rho(\lambda; \lambda_1, \lambda_2) = \rho_{\mathrm{vac}}(\lambda) + \frac{1}{N}(\sigma(\lambda - \lambda_1) + \sigma(\lambda - \lambda_2)) \tag{75}$$

where $\rho_{\mathrm{vac}}(\lambda)$ is the vacuum density of states and $\sigma(\lambda)$ is determined by the equation

$$\pi\sigma(\lambda) + \int_{-\infty}^{\infty} \frac{1}{1 + (\lambda - \mu)^2}\sigma(\mu)d\mu = -\pi\delta(\lambda). \tag{76}$$

The energy of the system is the sum of the vacuum energy plus the contribution from the holes

$$E = E_{\mathrm{vac}} + \mathcal{E}(l_1) + \mathcal{E}(l_2), \tag{77}$$

where

$$\mathcal{E}(\lambda) = \int_{-\infty}^{\infty} h(\mu)\sigma(\mu - \lambda)d\mu. \tag{78}$$

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Similarly, the relative momentum $k(\lambda)$ is given by

$$k(\lambda) = \int_{-\infty}^{\infty} p(\mu)\sigma(\mu - \lambda)d\mu. \tag{79}$$

It ranges from 0 to π. The relative energy $\mathcal{E}(\lambda)$ and momentum $k(\lambda)$ can be interpreted as the energy and momentum of the physical excitation. The dispersion relation is very simple

$$\mathcal{E}(k) = \frac{\pi}{2}\sin k. \tag{80}$$

The 1-particle excitation will come from the N-odd sector.

The scattering matrix S_{ex} for the triplet excitation is determined by the equation

$$\ln S_{\text{ex}}(\lambda) = \int \ln S(\mu)\sigma(\mu - \lambda)d\mu, \tag{81}$$

where $S(\mu)$ is given in (53). It is given by

$$S_{\text{ex}}(\lambda) = \frac{f(\lambda)}{f(-\lambda)}, \tag{82}$$

where

$$f(\lambda) = \frac{\Gamma(1/2 + i\lambda/2)}{\Gamma(1 + i\lambda/2)} = \frac{\Gamma(s)}{\Gamma(1/2 + s)}; \quad s = \frac{1}{2} + i\frac{\lambda}{2}. \tag{83}$$

It is interesting to note that the function $|f(\lambda)|^2$ appears also in the representations of $SL(2, R)$ as a Harish-Chandra factor.

The singlet solution can be described in a similar fashion, but one make use of the complex solution of BA equations (see [5]).

Up to now we have only considered the antiferromagnetic case (Hamiltonian negative definite). Let us briefly consider now the ferromagnetic case (Hamiltonian positive definite). The physical vacuum coincide with the Fock space vacuum state (compare with the antiferromagnetic case where the physical vacuum is constructed by filling the Dirac sea). In the infinite volume limit $N \to \infty$ equation (45) does not yield quantization of momentum. The energy (51) becomes positive after the change of sign of the Hamiltonian and corresponds to physical particles. The operators Σ_+ and Σ_- cease to exist and only

$$Q = \Sigma_3 - \frac{N}{2}, \tag{84}$$

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makes sense. Thus the $SU(2)$ symmetry breaks down. The spectrum has also bound states with the dispersion law

$$\mathcal{E}_M(p) = \frac{1}{2M+1}(1 - \cos p), \tag{85}$$

where M is an eigenvalue of charge Q.

Most of the techniques described so far apply to other integrable models as well. In general to define a model we need a group G, a representation ρ, and a spectral parameter. In our case the spectral parameter was defined on the complex plane. However, it may also be defined on a strip (non-degenerate rational case), or on a torus (elliptic case). We must introduce deformation parameters γ and κ to define these two cases. Thus one can classify all the integrable models in terms of the quartet

$$(G, \rho, \gamma, \kappa)$$

For example the NS model corresponds to general spin, whereas the Sine-Gordon to general spin plus one deformation.

3 Lecture 3

In this lecture we will consider higher spin generalizations of the spin-1/2 XXX model. Instead of spin-1/2 variables at each lattice site we now have spin-S variables. The local Hilbert space is modified accordingly

$$h_n = \mathrm{C}^{2S+1}. \tag{86}$$

One can define a Lax operator $L_n(\lambda)$ in the same way as earlier,

$$L_n(\lambda) = \begin{pmatrix} \lambda + is_n^3 & is_n^- \\ is_n^+ & \lambda - is_n^3 \end{pmatrix}, \tag{87}$$

but now acts on $\mathrm{C}^{2S+1} \otimes \mathrm{C}^2$. The commutation relations between the $L_{n,a}$'s are given by

$$R_{a_1,a_2}(\lambda - \mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu), \tag{88}$$
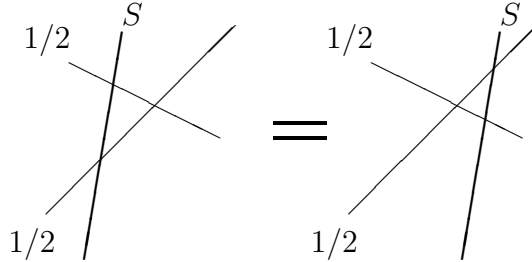
where R_{a_1,a_2} is given again by (20).

Figure 4. Graphical representation of (88). The thin lines represent "spin-1/2" lines, whereas the thick ones are "spin-S" lines.

The monodromy matrix $T(\lambda)$ is defined similarly with the spin-1/2 case. The vector $\Omega = \prod \otimes \omega_n$ is again an eigenvector of the operator $F(\lambda) = \mathrm{Tr}_a T(\lambda)$, where ω_n is a highest weight spin state

$$s_n^+ \omega_n = 0; \qquad s_n^3 \omega_n = S\omega_n. \tag{89}$$

Using again the same ansatz for the general eigenvector of $F(\lambda)$

$$\Phi(\{\lambda\}) = B(\lambda_1)B(\lambda_2)\cdots B(\lambda_l)\Omega, \tag{90}$$

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we get the Bethe equations

$$(\frac{\lambda_k + iS}{\lambda_k - iS})^N = \prod_{m=1;m\neq k}^{l} \frac{\lambda_k - \lambda_m + i}{\lambda_k - \lambda_m - i} \qquad k = 1, \ldots, l. \qquad (91)$$

If $\{\lambda\}$ is a solution of (91) then $\Phi(\{\lambda\})$ is an eigenvector of $F(\lambda)$. However, it is not clear at all how to get a local Hamiltonian. Since the auxiliary space is different from the local Hilbert space we can not use the same tricks we used in the spin-1/2 case where we had $L_{n,a} = P_{n,a}$ at some specific value of λ. In order to circumvent the problem we change the auxiliary space to $V = C^{2S+1}$. So, from a $S \otimes 1/2$ representation we go to a $S \otimes S$ representation. The new Lax operator $L_{n,f}$ is defined on $C^{2S+1} \otimes C^{2S+1}$ and is called the fundamental Lax operator[8]. This Lax operator is graphically represented by a cross of two S-lines (thick lines). In order to find its explicit form we have to solve simultaneously the equations which are graphically represented by the figures 5 and 6.
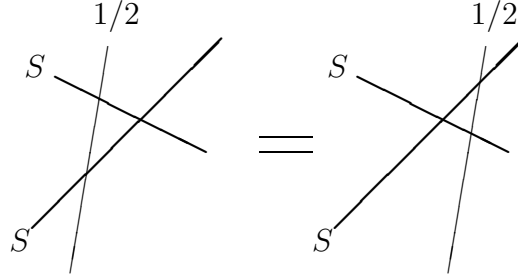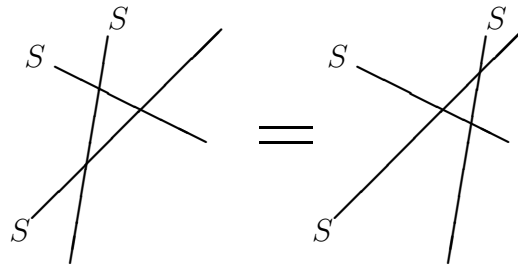

Figure 5. The S-S-1/2 equation.

Figure 6. The S-S-S equation.

Solution to these equation exists and it is unique. The fundamental mon-

odromy matrix is defined as

$$T_f = L_{n,f} \cdots L_{n,f}, \tag{92}$$

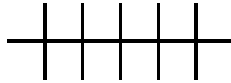and is represented as in figure 7.

Figure 7. Graphical representation of the fundamental monodromy matrix T_f.

We wish to prove that the trace of the fundamental monodromy matrix commutes with the trace of the auxiliary monodromy matrix

$$[\mathrm{Tr}_f T_f, \mathrm{Tr}_a T_a] = 0. \tag{93}$$

The first one will yield the local conservation laws whereas the second is the one which the Bethe ansatz gives. Having a solution of the equations described in figures 5 and 6, the "train argument" (fig. 8) immediately shows that (93) is satisfied.
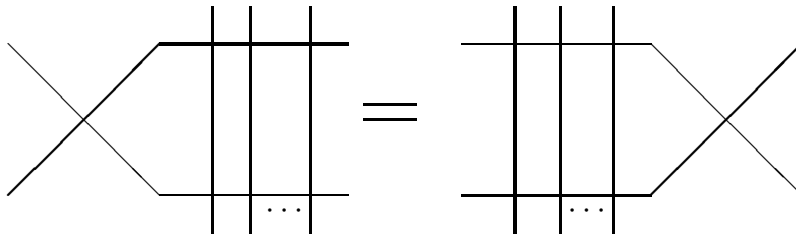

Figure 8. The "train argument" for equation (93).

Let us describe the algebraic structure underlying the Bethe ansatz. We propose the following mnemonic rule. Let us suppose that we have an asso-

ciative algebra $\mathcal{A}$, and an operator R on $\mathcal{A} \otimes \mathcal{A}$ which satisfies a universal Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{94}$$

defined on $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$. The indices indicate to which of $\mathcal{A}$'s the operator R belongs. For example R_{13} means that the operator R belongs to the first and third $\mathcal{A}$. Let $\rho(s, \lambda)$ be a representation of the algebra $\mathcal{A}$ characterized by a discrete index s (the spin in our case) and a complex variable λ (the spectral variable). A representative example is the loop algebra where a representation is given by $X_{a,n} = T^a \lambda^n$, where T^a are the spin matrices (however it is not our case). We define

$$(\rho(s_1, \lambda_1) \otimes \rho(s_2, \lambda_2))R = R^{s_1 s_2}(\lambda_1, \lambda_2). \tag{95}$$

We assume that $R^{s_1 s_2}(\lambda_1, \lambda_2)$ depends only on the difference $(\lambda_1 - \lambda_2)$. Consider the case $h_n = \mathrm{C}^{2S+1}, V = \mathrm{C}^2$, and apply (94) to

$$\rho(a_1, \lambda)\rho(a_2, \mu)\rho(n, \sigma), \tag{96}$$

where the a-indices refer to the auxiliary space and the n to the local Hilbert space,as usual. Equation (94) becomes

$$R_{a_1,a_2}(\lambda-\mu)R_{a_1,n}(\lambda-\sigma)R_{a_2,n}(\mu-\sigma) = R_{a_2,n}(\mu-\sigma)R_{a_1,n}(\lambda-\sigma)R_{a_1,a_2}(\lambda-\mu). \tag{97}$$

If we identify $R_{a,n}(\lambda)$ with $L_{n,a}(\lambda)$ then (97) is just equation (88). The value $\lambda = 0$ is assumed to be the specific point where $L_{n,a}$ coincides with the permutation operator $P_{n,a}$. We permute the indices in (94) according to the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

This yields

$$R_{12}R_{32}R_{31} = R_{31}R_{32}R_{12}. \tag{98}$$

Acting in (98) with

$$\rho(n_1, \lambda)\rho(n_2, \mu)\rho(a, \sigma), \tag{99}$$

we get

$$R_{n_1,n_2}(\lambda-\mu)R_{a,n_2}(\sigma-\mu)R_{a,n_1}(\sigma-\lambda) = R_{a,n_1}(\sigma-\lambda)R_{a,n_2}(\sigma-\mu)R_{n_1,n_2}(\lambda-\mu), \tag{100}$$

or with the identification of $R_{a,n}(\lambda)$ with $L_{n,a}(\lambda)$,

$$L_{n_1,a}(\sigma - \lambda)L_{n_2,a}(\sigma - \mu)R_{n_1,n_2}(\lambda - \mu) = R_{n_1,n_2}(\lambda - \mu)L_{n_2,a}(\sigma - \mu)L_{n_1,a}(\sigma - \lambda). \tag{101}$$

This is the exact form of the equation given graphically in figure 5.

It is now straightforward to find the Hamiltonian of a spin-S system. It is given by

$$H = \sum_n H_{n,n+1} = \sum_n f_s(\vec{s}_n \cdot \vec{s}_{n+1}), \tag{102}$$

where

$$H_{n_1,n_2} = \frac{1}{i}\frac{d}{d\lambda}R_{n_1,n_2}(\lambda)\Big|_{\lambda=0}P_{n_1,n_2}, \tag{103}$$

and $f_s(\vec{s}_n \cdot \vec{s}_{n+1})$ is polynomial of degree $2s$. For example for a spin-1 system we have

$$f_1(x) = x - x^2. \tag{104}$$

We now generalize our results to the case of the arbitrary complex spin variable S (Verma moduli). The spin matrices can be expressed in terms of Bose creation and annihilation operators ψ^* and ψ which satisfy

$$[\psi^*, \psi] = -1, \tag{105}$$

as (Holstein-Primakov)

$$s_n^+ = \psi_n^*(2S - \psi_n^*\psi_n)^{1/2}, \tag{106}$$

$$s_n^- = (2S - \psi_n^*\psi_n)^{1/2}\psi_n, \tag{107}$$

$$s_n^3 = \psi_n^*\psi_n - S. \tag{108}$$

If $\psi_n^*\psi_n$ is an integer then we are dealing with a finite subspace since necessarily

$$\psi_n^*\psi_n \leq 2S. \tag{109}$$

When S is complex s^+, s^- and s^3 form a representation of $SL(2,C)$ and not of $SU(2)$. In the limit $S \to \infty$ the spin matrices limit to

$$\lim_{s\to\infty} s_n^+ = \sqrt{2S}\psi_n^*, \tag{110}$$

$$\lim_{s\to\infty} s_n^- = \sqrt{2S}\psi_n, \tag{111}$$

$$\lim_{s\to\infty} s_n^3 = -S. \tag{112}$$

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Let us find the $S \to \infty$ limit of the Lax pair given in (87),

$$i\frac{L_n(\lambda)}{S}\sigma_3 = \frac{i}{S}\begin{pmatrix} \lambda + is_n^3 & is_n^- \\ is_n^+ & \lambda - is_n^3 \end{pmatrix}\sigma_3 \tag{113}$$

$$= I + \begin{pmatrix} i\lambda/S & \psi_n/\sqrt{2S} \\ -\psi_n^*/\sqrt{2S} & -i\lambda/S \end{pmatrix}. \tag{114}$$

Consider now the continuum limit $\Delta n \to x$. Let $\Delta = 1/S$. The creation and annihilation operators are of order $\sqrt{\Delta}$. To see this note that the discretized version of a delta function is

$$\delta(x - y) \to \frac{\delta_{mn}}{\Delta}, \tag{115}$$

so that (105) becomes

$$[\frac{\psi_n^*}{\sqrt{\Delta}}, \frac{\psi_m}{\sqrt{\Delta}}] = -\frac{\delta_{mn}}{\Delta}, \tag{116}$$

which proves our assertion. Defining the continuum ψ-field as

$$\psi(x) = \frac{\psi_m}{\sqrt{\Delta}}, \tag{117}$$

equation (114) becomes

$$\frac{iL(x, \lambda)}{S} = I + \Delta\begin{pmatrix} \lambda & \psi(x) \\ \psi^*(x) & -\lambda \end{pmatrix}, \tag{118}$$

where we have absorbed a factor of i in the parameter λ. But this exactly the Lax operator for nonlinear Schödinger (NS) equation[4]. Hence the NS model belongs to same class of model with the XXX magnetic chains. It is recovered in the $S \to \infty$ limit of the latter. The momentum and the energy in the NS model are given by k and k^2, respectively. Let us see if we can get these values from the XXX model. We have seen that the momentum in ferromagnetic case is given by

$$p(\lambda) = \frac{1}{i}\ln\Big(\frac{\lambda + iS}{\lambda - iS}\Big). \tag{119}$$

In the limit $S \to \infty$, $p(\lambda)$ goes as λ/S. Furthermore, the energy limits to

$$E = \frac{2S}{S^2 + \lambda^2} \sim \frac{2}{S} - \frac{2\lambda^2}{S^3}. \tag{120}$$

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Identifying k with an appropriately rescaled λ (and subtracting the constant from (120)) we see that the $S \to \infty$ limit of XXX model correctly reproduces the energy and momentum spectrum. Let us stress that here we differ from others, who prefer to view the NS model as a limit of the spin-1/2 XXZ model.

The spin-1/2 XXZ model is defined by the Hamiltonian

$$H = \sum_n (s_n^x s_{n+1}^x + s_n^y s_{n+1}^y + J s_n^z s_{n+1}^z), \tag{121}$$

where J is some constant different from 1. We will treat the XXZ model as a q-analogue of the XXX model.

The notion of q-deformation was introduced by Gauss in the last century. He defined the q-deformation of x as

$$x_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{122}$$

The limit $q \to 1$ gives back the original value

$$\lim_{q \to 1} x_q = x. \tag{123}$$

If we introduce a new variable γ,

$$q = e^{i\gamma}, \tag{124}$$

then x_q is given in terms of the new variable γ by

$$x_q = \frac{\sinh \gamma x}{\sin \gamma}. \tag{125}$$

One can now deform the Lax operator of the XXX model as

$$L(\lambda, \gamma) = \begin{pmatrix} \sinh[\gamma(\lambda + is^3)] & is^- \sin \gamma \\ is^+ \sin \gamma & \sinh[\gamma(\lambda - is^3)] \end{pmatrix}. \tag{126}$$

The deformed R matrix looks as follows

$$R = \begin{pmatrix} \sinh[\gamma(\lambda + i)] & 0 & 0 & 0 \\ 0 & \sinh \gamma\lambda & i \sin \gamma & 0 \\ 0 & i \sin \gamma & \sinh \gamma\lambda & 0 \\ 0 & 0 & 0 & \sinh[\gamma(\lambda + i)] \end{pmatrix}. \tag{127}$$

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Note that we have changed the normalization w.r.t. (20). If we impose the fundamental commutation relations (19), we discover that the spin matrices obey deformed algebraic relations[9],

$$[s^3, s^\pm] = \pm s^\pm \tag{128}$$

$$[s^+, s^-] = \frac{\sin(2\gamma s^3)}{\sin \gamma}. \tag{129}$$

This algebraic relations define what we now call quantum group. Actually, a better name would be deformed Lie algebra. In the mathematical literature the notion of deformation of algebraic structure is well defined. One starts by making the structure constants $C_{ij...}$ t-dependent, where t is a deformation parameter. The algebraic structure determined by the deformed structure constants is, in general, nonequivalent to the original one since some of the properties of the corresponding algebra are changed after deformation. A well known example is quantum mechanics. It has been proven that quantum mechanics is stable deformation of classical mechanics (the deformation variable is the $\hbar$). In this sense, one can use the word "quantum" instead of the word "deformation". Drin'field was the first who used the term "quantum" in this sense and thus coined the term "quantum group" [10].

4   Lecture 4

In the last lecture we have seen that the XXZ_S model can be viewed as a q-deformation of the XXX_S model. We have constructed the Lax operator $L(\lambda)$ of the XXZ model as a q-deformation of Lax operator of the XXX model and we have seen that it obeys the fundamental commutation relations,

$$R(\lambda - \mu)L^1(\lambda)L^2(\mu) = L^2(\mu)L^1(\lambda)R(\lambda - \mu), \qquad (130)$$

only when the spin matrices obey certain relations. These considerations led to the discovery of the quantum groups.

Now we shall elaborate on this in more detail. It will be convenient to perform a similarity transformation in all entries in (130)

$$L(\lambda) \rightarrow Q(\lambda)L(\lambda)Q^{-1}(\lambda) \qquad (131)$$

and

$$R(\lambda - \mu) \rightarrow Q^1(\lambda)Q^2(\mu)R(\lambda - \mu)Q^1(\lambda)^{-1}Q^2(\mu)^{-1}, \qquad (132)$$

where

$$Q(\lambda) = \begin{pmatrix} q^{-i\lambda/2} & 0 \\ 0 & q^{i\lambda/2} \end{pmatrix}. \qquad (133)$$

In matrix form $Q^1(\lambda)Q^2(\mu)$ is given by

$$Q^1(\lambda)Q^2(\mu) = \begin{pmatrix} q^{-i(\lambda+\mu)/2} & 0 & 0 & 0 \\ 0 & q^{-i(\lambda-\mu)/2} & 0 & 0 \\ 0 & 0 & q^{i(\lambda-\mu)/2} & 0 \\ 0 & 0 & 0 & q^{i(\lambda+\mu)/2} \end{pmatrix}, \qquad (134)$$

and only the middle block of it acts non-trivially in (132). So the new R-matrix depends on the difference $(\lambda - \mu)$. Explicitly we have now

$$L(\lambda, \gamma) = \begin{pmatrix} \sinh[\gamma(\lambda + is^3)] & is^- \sin \gamma e^{\lambda\gamma} \\ is^+ \sin \gamma e^{-\lambda\gamma} & \sinh[\gamma(\lambda - is^3)] \end{pmatrix}, \qquad (135)$$

and,

$$R = \begin{pmatrix} \sinh[\gamma(\lambda + i)] & 0 & 0 & 0 \\ 0 & \sinh \gamma\lambda & i \sin \gamma e^{\lambda\gamma} & 0 \\ 0 & i \sin \gamma e^{-\lambda\gamma} & \sinh \gamma\lambda & 0 \\ 0 & 0 & 0 & \sinh[\gamma(\lambda + i)] \end{pmatrix}. \qquad (136)$$

We introduce a new variable

$$x = \exp \gamma \lambda. \tag{137}$$

In terms of this variable the Lax operator decomposes as follows

$$L = xL_+ - \frac{1}{x}L_-, \tag{138}$$

where

$$L_+ = \begin{pmatrix} q^{H/2} & (q - q^{-1})s^- \\ 0 & q^{-H/2} \end{pmatrix}, \tag{139}$$

$$L_- = \begin{pmatrix} q^{H/2} & 0 \\ -(q - q^{-1})s^+ & q^{-H/2} \end{pmatrix}, \tag{140}$$

and we introduce a notation $q^{H/2} = \exp i\gamma s^3$. Similarly, the R-matrix can be written as

$$R(\lambda) = xR_+ - \frac{1}{x}R_-, \tag{141}$$

where

$$R_+ = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & (q - q^{-1}) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{142}$$

and,

$$R_- = PR_+^{-1}P. \tag{143}$$

P is the permutation operator. We now substitute (138) and (141) into (130). In each side of the resulting equation seven different powers of x and y ($y = \exp \gamma \mu$) appear, namely

$$\{x^2, y^2, x^2y^2, x^2/y^2, y^2/x^2, 1/(x^2y^2), 1\}.$$

This implies seven different equations which relate $R_\pm$ with $L_\pm$. So, for example, equating the coefficients of "1" we get

$$R_+ L_-^1 L_+^2 + R_- L_+^1 L_-^2 = L_+^2 L_-^1 R_+ + L_-^2 L_+^1 R_-. \tag{144}$$

The other six equations are given by

$$R_{\pm(\mp)}L_{\pm}^1 L_{\pm}^2 = L_{\pm}^2 L_{\pm}^1 R_{\pm(\mp)} \tag{145}$$
$$R_+ L_+^1 L_-^2 = L_-^2 L_+^1 R_+ \tag{146}$$
$$R_- L_-^1 L_+^2 = L_+^2 L_-^1 R_-, \tag{147}$$

where the notation $R_{\pm(\mp)}$ means that we have to consider all four cases, namely plus or minus in R, plus sign in both L's, and plus or minus in R, minus sign in both L's. So, equation (145) is, in fact, four equations. However, only three of the seven equations are independent. We choose as our independent equations the following three

$$RL_+^1 L_+^2 = L_+^2 L_+^1 R, \tag{148}$$
$$RL_-^1 L_-^2 = L_-^2 L_-^1 R, \tag{149}$$
$$R_+ L_+^1 L_-^2 = L_-^2 L_+^1 R_+, \tag{150}$$

where R can be either R_+ or R_-. Let us illustrate how we can derive the rest of the equations starting from the above three. In particular we derive the following equation

$$R_- L_+^1 L_+^2 = L_+^2 L_+^1 R_-, \tag{151}$$

starting from the equation

$$R_+ L_+^1 L_+^2 = L_+^2 L_+^1 R_+. \tag{152}$$

We multiply both sides with the permutation operator P. The left hand side gives

$$PR_+ L_+^1 L_+^2 = R_-^{-1} P L_+^1 L_+^2$$
$$= R_-^{-1} L_+^2 L_+^1 P, \tag{153}$$

where in the first step we used (143). Similarly, the right hand side yields

$$PL_+^2 L_+^1 R_+ = L_+^1 L_+^2 R_-^{-1} P. \tag{154}$$

Equations (153) and (154) imply (151). All but equation (144) can be derived similarly. Equation (144) can be checked using the so-called Hecke property

$$R_+ - R_- = (q - \frac{1}{q})P, \tag{155}$$

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which can be checked explicitly from (142) and (143).

The relations (145)-(147) can be taken as the defining relations for the q-deformation of the Lie algebra $SL(2)$ expressed in terms of its generators q^H, s_+ and s_-, entering in (139), (140)[11]. They are homogeneously quadratic. The matrices $R_\pm$ play the role of the corresponding structure constants. The Yang-Baxter (Y-B) relation

$$R_+^{12} R_+^{13} R_+^{23} = R_+^{23} R_+^{13} R_+^{12}, \qquad (156)$$

and analogous relations involving R_- guarantee that higher order relations follow from the quadratic ones. (One can derive (156) from (97) for $n = a_3$ in an exactly the same way as we find (145)-(147) from (130)). The relation (156) guarantees that (148)-(150) are the only relations defining the deformed algebra $SL(2)_q$. Indeed, following the two paths in the diagram

$$L^1 L^2 L^3 = \begin{array}{c} L^1 L^3 L^2 \to L^3 L^1 L^2 \\ \nearrow \qquad\qquad \searrow \\ \searrow \qquad\qquad \nearrow \\ L^2 L^1 L^3 \to L^2 L^3 L^1 \end{array} = L^3 L^2 L^3. \qquad (157)$$

we get the cubic relation

$$R^{123} L^3 L^2 L^1 (R^{123})^{-1} = R^{321} L^3 L^2 L^1 (R^{321})^{-1}, \qquad (158)$$

where R^{123} and R^{321} are the left hand side and the right hand side of the relation (156), respectively. So, due to this relation the relation (158) is empty. Very general theory-categorical considerations show that if the cubic relations are absent (so if the Y-B relation (156) holds) then no new relations will occur in any order of L.

Consider now the operator

$$\hat{R} = P R_+. \qquad (159)$$

The Yang-Baxter relation takes the form

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}, \qquad (160)$$

and the Hecke relation is

$$\hat{R}^2 = I + (q - \frac{1}{q}) \hat{R}. \qquad (161)$$

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This is a q-deformation for the transposition generators of the symmetric group σ^{ik}, realized through the permutation operator P, i.e.,

$$P_{12}P_{23}P_{12} = P_{23}P_{12}P_{23}, \qquad (162)$$

and

$$P_{ik}^2 = I. \qquad (163)$$

Indeed, we have

$$\hat{R}\Big|_{q=1} = P, \qquad (164)$$

so that (162) and (163) are the $q \to 1$ limit of (160) and (161), respectively.

The Y-B relations (156) are characteristic of the Braid group of Artin. The representation (135) which we associate with the Lie algebra $SL(2)_q$ can be generalized to $SL(N)_q$. The R-matrix will be then a triangular matrix $N^2 \times N^2$ and the Hecke relation still holds. Other classical Lie algebras lead to their own R-matrix and some generalization of the Hecke relation.

The $SL(N)$ q-deformed Lie algebra is defined in terms of the corresponding R-matrix by the relations (145)-(147), where matrices of generators $L_\pm$ are triangular and the diagonal elements are arranged in such a way that

$$l_{ii}^+ l_{(N-i),(N-i)}^- = 1. \qquad (165)$$

This gives the definition of the q-deformed Lie algebra $SL(N)_q$ (or A_{N-1}) corresponding to $SL(N)$. One can give a similar definition of the q-deformed Lie algebras corresponding to all classical series B, C and D, see [11].

After this general discussion we return to the $SL(2)_q$ algebra. We redefine R_+ by multiplying it with $q^{-1/2}$,

$$R_+ \to q^{-1/2} R_+ = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & (q - q^{-1})q^{-1/2} & 0 \\ 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix}, \qquad (166)$$

and consider its blocks as the spin $1/2$ representation of the matrix L_+. We see that the q-deformation of the Pauli matrices are given by

$$q^{H/2} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \qquad (167)$$

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$$s^- = \begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{pmatrix}, \tag{168}$$

and,

$$s^+ = \begin{pmatrix} 0 & q^{-1/2} \\ 0 & 0 \end{pmatrix}. \tag{169}$$

Let us note that here the structure constants (R-matrix) appear in the fundamental representation rather than in the adjoint one.

Important property of quantum group is a comultiplication operation, which corresponds to the addition of spins in the $q = 1$ limit. Let $L'_\pm$ and $L''_\pm$ be two independent (commuting) set of generators. Then their matrix products $L'_\pm L''_\pm$ satisfy the same relation as each of the L's.. Indeed, we have

$$R(L'_+ L''_+)^1 (L'_+ L''_+)^2 = (L'_+ L''_+)^2 (L'_+ L''_+)^1 R. \tag{170}$$

One can combine the generators of quantum group into one matrix instead of two, introducing a matrix L,

$$L = L_+ L_-^{-1}. \tag{171}$$

It satisfies the equation

$$L^1 R_-^{-1} L^2 R_- = R_+^{-1} L^2 R_+ L^1. \tag{172}$$

This equation can be proven by using manipulations similar to the ones we have used earlier. Let us see, for example, how the left hand side arises. We start from equation (146) and multiply from the left with $L^1 R_-^{-1}$ and from the right with $R_-^{-1}(L_-^1)^{-1}(L_-^2)^{-1}R_-$. Then the right hand side of (146) yields the left hand side of (172). The relation (171) is a kind of Gauss decomposition for the matrix L, since the diagonal of L is divided by (165).

The classical limit $q \to 1$ or equivalently $\gamma \to 0$ is more transparent in terms of L. The R_+ matrix becomes for small γ

$$R_+ = I + i\gamma r_+ + \mathcal{O}(\gamma^2), \tag{173}$$

where

$$r_+ = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 2 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{174}$$

Similarly, one can define a matrix r_- from the expansion of R_-. Then the difference between r_+ and r_- is a Casimir operator,

$$C \equiv r_+ - r_- = \vec{\sigma} \otimes \vec{\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (175)$$

We expand also the matrix L,

$$L = I + \gamma l + \mathcal{O}(\gamma^2). \qquad (176)$$

Let us keep the Planck constant $\hbar$, so that $q = e^{i\gamma\hbar}$. The fundamental commutation relations imply that

$$l^1 l^2 - l^2 l^1 = \hbar [C, l^2]. \qquad (177)$$

It is evident, that we got the relations of Lie algebra $SL(2)$ written in terms of the "structure constant" C. The usual form

$$[l^a, l^b] = i\hbar\epsilon^{abc}l^c, \qquad (178)$$

is obtained if one introduces l^a as

$$l = \sum_a l^a \sigma_a. \qquad (179)$$

In terms of the universal enveloping algebra, generated by l^a or $L_\pm$, we can say that it was the comultiplication which was deformed and the multiplication, which was left intact.

The opposite deformation takes place for the dual object, q deformation Lie group, corresponding to a given Lie algebra. Let us deformed it also in terms of matrix elements - coordinates on the group manifold - $T = \|t_{ij}\|$. The following relation

$$RT^1 T^2 = T^2 T^1 R, \qquad (180)$$

make coordinates non-commutative, introducing a new non-commutative multiplication. The comultiplication

$$T', T'' \to T = T'T'', \qquad (181)$$

is the same as in the classical case $q = 1$.

It is possible to combine the generators L and T to define a bigger algebra. For that the following relation between T and L suffices

$$T^2 L^1 = R_+ L^1 R_-^{-1} T^2. \tag{182}$$

which corresponds to the left action of Lie algebra on its Lie group. The system of equations (172), (180) and (182) defines the cotangent bundle $(T^*G)_q$. In the semi-classical limit we have

$$[T^2, l^1] = C T^2. \tag{183}$$

The set of relations (172), (180) and (182) is covariant with respect to the left shifts

$$T \to TS, \quad L \to SLS^{-1}, \tag{184}$$

but then S is necessarily quantized, namely it obeys

$$R S^1 S^2 = S^2 S^1 R. \tag{185}$$

Note the matrix elements of S commute with the ones of L and T.

5 Lecture 5

Let us describe in more detail the system of relations (172), (180) and (182) which defines a "quantum top". As we will see later, it is the quantum top which drives the zero mode of the WZNW model. The phase space of the classical top consists of coordinates g which are group elements and momenta ω which are elements of the corresponding Lie algebra. The Lagrangian of the system is given by

$$L = \mathrm{Tr}\,\dot{g}g^{-1}\omega - \frac{1}{2}\mathrm{Tr}\,\omega^2, \tag{186}$$

and the basic Poisson brackets are

$$\begin{aligned}
\{g^1, g^2\} &= 0, \tag{187}\\
\{\omega^1, \omega^2\} &= [C, \omega^2], \tag{188}\\
\{\omega^1, g^2\} &= Cg^2, \tag{189}
\end{aligned}$$

where $C = \vec{\sigma} \otimes \vec{\sigma}$ and, as before

$$\begin{aligned}
g^1 &= g \otimes I \tag{190}\\
g^2 &= I \otimes g \quad \text{etc.} \tag{191}
\end{aligned}$$

Using the equations of motion

$$\begin{aligned}
\dot{\omega} &= 0 \tag{192}\\
\dot{g} &= \omega g \tag{193}
\end{aligned}$$

we get for the time development of the system

$$g(t) = \exp[\omega(0)t]g(0). \tag{194}$$

In the quantum case our system of commutation relations from the Lecture 4 is

$$\begin{aligned}
L^1 R_-^{-1} L^2 R_- &= R_+^{-1} L^2 R_+ L^1, \tag{195}\\
RT^1 L^2 &= T^2 T^1 R, \tag{196}\\
T^2 L^1 &= R_+ L^1 R_-^{-1} T^2. \tag{197}
\end{aligned}$$

In the limit $\gamma \to 0$ we identify g with T and ω with l, where l is the first term in the γ expansion of L,

$$L = I + \gamma l + \cdots. \tag{198}$$

Then the quantum commutation relations (195), (196) and (197) limit to (187), (188) and (189), respectively. ($C = r_+ - r_-$, see eq.(175)). The time evolution of the system is described by the equation

$$T(n) = L^n T(0), \tag{199}$$

where the discrete index n plays the role of time. The interpretation of $T(n)$ as a time-evolved coordinate is supported by the fact that the pair $(T(n), L)$ obeys the same quantum commutation relations as the pair $(T(0), L)$.

Let us now consider the Wess-Zumino-Novikov-Witten (WZNW) model. The degree's of freedom are group elements $g(x, t)$ and the left and right currents $j_\mu^L = \partial_\mu g g^{-1}$ and $j_\mu^R = \partial_\mu g^{-1} g$, respectively. The model is defined on a cylinder $R^1 \otimes S^1$ or the sphere S^2 depending whether we are in Minkowski or Euclidean picture. The left and right currents separately generate a Kac-Moody algebra. These are two independent algebras since left currents commute right currents. From now on we restrict our attention to the left sector. Let

$$l = j_0 + j_1. \tag{200}$$

Then the fact that l generates a Kac-Moody algebra is expressed through the following Poisson bracket between the l's

$$\{l^1(x), l^2(y)\} = \gamma [l^2(y), C] \delta(x - y) + \gamma C \delta'(x - y), \tag{201}$$

where γ is the coupling constant and is related to the level k_{cl} of the Kac-Moody algebra by

$$k_{\text{cl}} = \frac{\pi}{\gamma}. \tag{202}$$

Here we write k_{cl} because quantum mechanically the level might renormalize.

The quantum lattice picture for this looks as follows. Consider a chain with an operator L_n attached to each site. Let $R_{\pm(n-m)}$ be

$$R_{\pm(n-m)} = \begin{cases} I & \text{if } n \neq m \\ R_\pm & \text{if } n = m \end{cases} \tag{203}$$

We impose the following set of commutation relations for L_n

$$L_n^1 R_{-(n-m-1)}^{-1} L_m^2 R_{-(n-m)} = R_{+(n-m)}^{-1} L_m^2 R_{+(n-m+1)} L_n^1. \tag{204}$$

It is easy to see that the shift of arguments in two R's generates a derivative of a delta function in the classical limit. Indeed for $\hbar \to 0$, $\Delta \to 0$

$$R_{\pm(m-n)} \to I + \gamma \delta_{mn} r_\pm, \tag{205}$$

and,

$$L_n \to I + \Delta l(x). \tag{206}$$

(Recall that the Lax operator is a kind of connection on the 1D lattice, so

$$L_n \sim \overleftarrow{\exp} \int_\Delta l(x) dx, \tag{207}$$

where $\overleftarrow{\exp} \int$ means a path ordered integral.)
Furthermore,

$$\frac{1}{\Delta} \delta_{mn} \to \delta(x - y), \tag{208}$$

and,

$$\frac{1}{\Delta^2} (\delta_{n,m+1} - \delta_{nm}) \to \delta'(x - y). \tag{209}$$

Combining the terms of order Δ^2 in (204) we get (201) with the δ' term present. Quantum corrections make the following connection of deformation parameter q and Kac-Moody level k:

$$q = \exp \frac{i\pi}{k+2}. \tag{210}$$

Equation (204) is, in fact, three equations

$$L_n^1 L_n^2 R_- = R_+^{-1} L_n^2 L_n^1 \quad (m = n), \tag{211}$$
$$L_n^1 L_{n+1}^2 = L_{n+1}^2 R_+ L_n^1 \quad (m = n+1), \tag{212}$$

$$[L_n^1, L_m^2] = 0 \quad \text{for } |m - n| \ge 2. \tag{213}$$

It is amazing that the our system still "remembers" the Virasoro algebra. Indeed, let $\mathcal{A}_N$ be the quantum structure generated by the Lax operators L_n in a lattice with N lattice sites. Then one can consider a new quantum

structure $\mathcal{A}_{N'}$ generated by products of some neighboring L's, where $N' < N$. Then $\mathcal{A}_{N'} \hookrightarrow \mathcal{A}_N$. In other words if, for example, L_1, L_2, L_3, L_4, L_5 obey the commutation relations (204) so does the set $L_1' = L_1 L_2$, $L_2' = L_3$ and $L_3' = L_4 L_5$. Hence, the density of points in the lattice is irrelevant. This is a kind of reparametrization invariance in the lattice.

Consider now the local field $g(x)$ for the WZNW model. In terms of the holonomy

$$u(x) = \overleftarrow{\exp} \int_0^x l(x) dx, \tag{214}$$

and the corresponding holonomy for the right current $v(x)$ it can be expressed as

$$g(x) = v(x) g(0) u(x), \tag{215}$$

where $g(0)$ is a zero mode of the local field. We want to find the lattice analogue of this expression. We introduce two vertex operator u_n and v_n,

$$u_n = L_n L_{n-1} \cdots L_1, \tag{216}$$

and an analogue expression for v_n made from right sector L's. Then,

$$g_n = v_n g(0) u_n. \tag{217}$$

The vertex operator u_N is actually equal to the monodromy matrix $M_N = L_N L_{N-1} \cdots L_1$. It satisfies the fundamental commutation relations

$$M^1 R_-^{-1} M^2 R_- = R_+^{-1} M^2 R_+ M^1. \tag{218}$$

which coincides with that of the q-deformed algebra (195). The derivation follows from the algebra (211)-(213). The commutation relations (218) does not depend on the lattice spacing Δ and presumably stay intact in the proper continuum limit. Thus the quantum Lie group naturally enters the conformal field theory as a monodromy of the local current.

It can be shown now, that the local field g_n is commutative

$$g_n^1 g_n^2 = g_n^2 g_n^1; \quad n \neq m, \tag{219}$$

and periodic

$$g_{n+N} = g_n, \tag{220}$$

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if the pair $(g(0), M)$ constitutes the top, i.e. the following commutation relations hold

$$Rg^1(0)g^2(0) = g^2(0)g^1(0)R, \qquad (221)$$

and

$$g^2(0)M^1 = R_+ M^1 R_-^{-1} g^2(0). \qquad (222)$$

This means, that the q-deformed top dynamical system is a complete system of zero modes of WZNW model and the spectrum of this latter field-theoretical models patterned by the spectrum of the former finite-dimensional. Indeed, the Hilbert space of the WZNW model can be written as

$$\mathcal{H}_{\text{WZNW}} = \sum_j \mathcal{H}_j \otimes \mathcal{H}_j, \qquad (223)$$

which is a sum of tensor squares of the irreducible representation $\mathcal{H}_j$ of the Kac-Moody algebra of spin j, $j = 0, 1/2, \ldots k/2$. An analogue formula holds for the Hilbert space of the top, namely

$$\mathcal{H}_{\text{top}} = \sum_j V_j \otimes V_j, \qquad (224)$$

where V_j are finite dimensional representations of the q-deformed Lie algebra $SL(2)_q$. Moreover the embedding of $SL(2)_q$ into Kac-Moody described above allows to state, that

$$\mathcal{H}_j = V_j \otimes \mathcal{H}_0, \qquad (225)$$

thus separating the contribution of the zero modes and oscillator degrees of freedom. This formula gives natural definition of fusion rules

$$\mathcal{H}_i \hat{\otimes} \mathcal{H}_j = V_i \otimes V_j \otimes \mathcal{H}_0. \qquad (226)$$

Note that the factor $\mathcal{H}_0$ appears only once. These fusion rules are based on a new comultiplication law between the Kac-Moody and the quantum algebra[12].

Let us illustrate the relations among the various algebras generated by a single Lie algebra g with a diagram:

$$
\begin{array}{ccccccc}
 & & & g & & & \text{Level 0} \\
 & & \swarrow & & \searrow & & \\
CS & \sim & KM_k & & & g_q & \text{Level 1} \\
 & \swarrow & & \searrow & \swarrow & & \searrow \\
? & & & KM_{k,q} & & g_{q,k} & \text{Level 2}
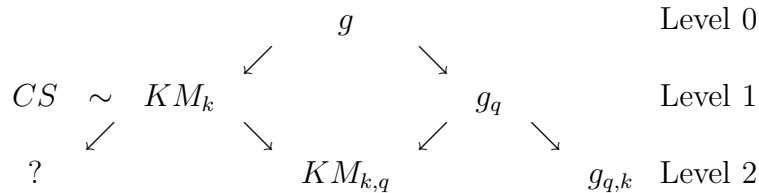\end{array}
$$

Figure 9. Relations between the algebras generated by the Lie algebra g.

At the level 1 we have either an affine infinite dimensional algebra algebra KM_k or a finite dimensional but deformed algebra g_q. Both of them are parametrized by one parameter, namely the level k for the Kac-Moody algebra and the deformation parameter q for the quantum algebra. As we have seen these two algebras are intimately related. Furthermore, the Kac-Moody algebra is related with a Chern-Simon theory (CS). At the level 2 we can have a local Lie algebra in 2 dimensions, an infinite deformed Lie algebra, or a further deformed finite dimensional quantum algebra. For the last two options we have natural candidates, namely the deformed Kac-Moody algebra $KM_{k,q}$ and the elliptic Sklyanin algebra $g_{q,k}$, respectively. However, the relation between them has not been completely understood yet[13]. For the first option there exist no natural candidate yet. Maybe a double-loop algebra is a possible candidate but this not clear at the moment. We stop our speculations here.

Let us now briefly show how massive models fit into our framework. We consider the Sine-Gordon model (SG) for concreteness. The Lax operator is the one of the XXZ model:

$$L(\lambda, \gamma) = \frac{1}{\sin \gamma} \begin{pmatrix} \sinh[\gamma(\lambda + is^3)] & is^- \sin \gamma \\ is^+ \sin \gamma & \sinh[\gamma(\lambda - is^3)] \end{pmatrix}. \qquad (227)$$

However, the realization of $s^\pm, s^3$ is different. We parametrize them in terms of real canonical fields π_n and ϕ_n,

$$[\pi_n, \phi_n] = -iI. \qquad (228)$$

The explicit form of the realization is the following

$$s^- = \frac{1}{2\kappa \sin \gamma} e^{-i\pi/2}(1 + \kappa^2 e^{-2i\phi})e^{-i\pi/2}, \qquad (229)$$

$$s^+ = (s^-)^\dagger, \qquad (230)$$

$$s^3 = \phi. \qquad (231)$$

It follows that the fundamental commutation relations are satisfied. In the limit $\Delta \to 0$ and with $\kappa = m\Delta$ one recovers the Lax operator for the Sine-Gordon model with mass m. Therefore, the SG model really belongs to the same class with the XXZ models which in turn is a deformation of the XXX model. Here we again differ from others who have studied the SG starting from the XYZ model of spin $1/2$.

However, if we try to repeat the same analysis as we did for the XXX or the XXZ model we run into a problem. The lower left matrix element of L_n has no zero eigenvalue. To get around this problem we consider the product of two Lax operators

$$L_n L_{n-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{232}$$

Indeed, $L_n L_{n-1}$ becomes an upper diagonal matrix when applied to a suitable chosen vacuum state Ω. Furthermore, the state Ω is an eigenvector of A and D,

$$
\begin{aligned}
A\Omega &= a(\lambda)\Omega, & (233) \\
D\Omega &= d(\lambda)\Omega & (234)
\end{aligned}
$$

The Bethe equation reads

$$
\begin{aligned}
\left(\frac{a(\lambda)}{d(\lambda)}\right)^{N/2} &= \left(\frac{\cosh(\lambda + \omega - i\gamma/2)\cosh(\lambda - \omega - i\gamma/2)}{\cosh(\lambda + \omega + i\gamma/2)\cosh(\lambda - \omega - i\gamma/2)}\right)^{N/2} & (235) \\
&= \prod_\mu \frac{\sinh(\lambda - \mu + i\gamma)}{\sinh(\lambda - \mu - i\gamma)}, & (236)
\end{aligned}
$$

where ω satisfies the equation

$$2\cosh 2\omega = m^2 + \frac{1}{m^2}. \tag{237}$$

Let us compare this result with the Bethe equation for the XXZ model. There the left hand side has the form

$$\frac{\sinh(\lambda + i\gamma S/2)}{\sinh(\lambda - i\gamma S/2)}$$

The hyperbolic cosines in (235) cause no problem because we can shift the λ's to convert them to hyperbolic sines. The right hand side remains unchanged since it depends only on differences of λ's. We conclude that the Sine-Gordon model can be viewed as inhomogeneous spin (-1/2) chain.

Up to now we have consider only discretization in the space direction. One can also consider discretization in the time direction. There we have light-like Lax operators $\hat{L}$ which connect the lattice points in the light-like direction (Fig. 10).
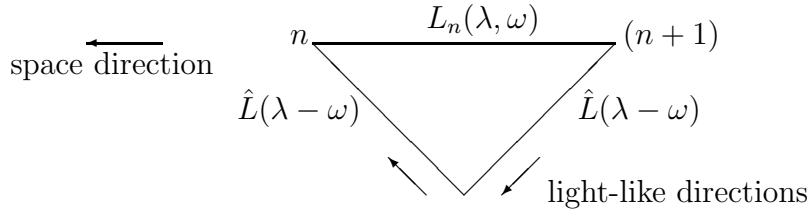
Figure 10. Discretization in the time direction.

For further discussion we refer to the article [14].

This concludes these lectures. We hope, that we were able to show, that the magnetic chains and the Bethe Ansatz contain a lot of potential for classifying and solving the integrable models of quantum field theory. The parameters, entering the description of magnetic chain, i.e. group G, its representation and one (or two) anisotropy parameters can be identified with dynamical field variables, coupling constants and mass parameter. All known integrable models could be put into this scheme. The models of conformal field theory then appear as a particular (massless) limit.

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