Zero modes in the Liouville model

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June 2013

Liouville model – CFT of rang 1

$$\phi(x,t), \quad \mathbb{S}^1 \times \mathbb{R}, \quad \phi(x+2\pi,t) = \phi(x,t)$$

$$\phi_{tt} - \phi_{xx} + e^{2\phi} = 0$$

$$H = \frac{1}{2\gamma}\int_0^{2\pi}(\pi^2+\phi_x^2+e^{2\phi})\,dx, \quad \{\pi(x),\phi(y)\} = \gamma\delta(x-y)$$

Zero curvature presentation

$$\widehat{L}_1 = \frac{d}{dx} - L_1, \quad \widehat{L}_0 = \frac{d}{dt} - L_0$$

$$L_1 = \frac{1}{2}\begin{pmatrix}\phi_t & e^{\phi} \\ e^{\phi} & -\phi_t\end{pmatrix}, \quad L_0 = \frac{1}{2}\begin{pmatrix}\phi_x & -e^{\phi} \\ e^{\phi} & -\phi_x\end{pmatrix}$$

$$[\widehat{L}_1, \widehat{L}_0] = 0$$

Free field realization via chiral fields

$$\chi(x) = q + p\chi + \chi_0(x), \quad q, p + \text{oscillators}$$

Reflection $p \to -p$

Origine?

Holonomy $T_x = L_1 T$, $T_t = L_0 T$, $T(0,0) = I$

$$T(x,t) = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & D(x,t) \end{pmatrix}$$

$$f(x,t) = \frac{A(x,t)}{B(x,t)} = f(x-t), \quad g(x,t) = \frac{C(x,t)}{D(x,t)} = g(x+t)$$

$$f(0) = \infty, \quad g(0) = 0$$

$$e^{2\phi} = -4 \frac{f'(x-t)g'(x+t)}{(f(x-t) - g(x+t))^2}$$

$$T(2\pi, 0) = M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ monodromy for } t = 0$$

$$f(x + 2\pi) = M(f(x)) = \frac{af + c}{bf + d}, \quad g(x + 2\pi) = M(g(x))$$

Invariants − tr $M = a+d$ and fixed points ξ, η − solutions of

$$b\xi^2 - (a + d)\xi - c = 0$$

$$MN = ND$$

$$N = \begin{pmatrix} 1 & -1/\xi \\ -\eta & 1 \end{pmatrix}, \quad D = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

$$M = \frac{1}{\xi - \eta} \begin{pmatrix} \xi\lambda - \eta\lambda^{-1} & \lambda - \lambda^{-1} \\ -\xi\eta(\lambda - \lambda^{-1}) & \xi\lambda^{-1} - \eta\lambda \end{pmatrix}$$

Poisson structure (Takhtajan, L.F. 1984)

$$\{a,b\} = \gamma ab, \quad \{a,c\} = \gamma ac, \quad \{d,b\} = -\gamma db,$$
$$\{d,c\} = -\gamma dc, \quad \{a,d\} = 2\gamma bc, \quad \{b,c\} = 0$$

$$\alpha^2 = b/c \text{ – central element}$$

$$\lambda = e^P, \quad \xi = \alpha e^Q, \quad \eta = -\alpha e^{-Q}$$

$$\{P,Q\} = \gamma, \quad \{\alpha, P\} = 0, \quad \{\alpha, Q\} = 0$$

$$\widehat{f} = N(f) = \frac{f - \eta}{-f/\xi + 1}$$

$$\widehat{f}(0) = -\xi = -\alpha e^{Q}, \quad \widehat{f}(2\pi) = e^{P}\widehat{f}(0)$$

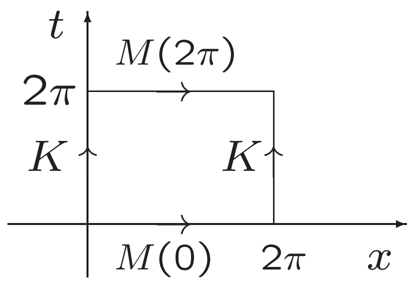
$$\ln \widehat{f}(x)/-\alpha = Q + Px/2\pi + \chi(x), \quad \chi(x) - \text{periodic}$$

Reflection $P \to -P$ – Weyl reflection

Convention $P > 0$

$$M = \frac{1}{\mathrm{Ch}\, Q}\begin{pmatrix} \mathrm{Ch}(Q+P) & \alpha\, \mathrm{Sh}\, P \\ \alpha^{-1}\, \mathrm{Sh}\, P & \mathrm{Ch}(Q-P) \end{pmatrix}$$

Evolution (Volkov, L.F. 2008)



$$M(2\pi) = K^{-1} M(0) K$$

K can be expressed via a, b, c, d

$$Z_n = \frac{\mathrm{Ch}(Q + nP)}{\mathrm{Sh}\, P}, \qquad M = \begin{pmatrix} \frac{Z_0}{Z_{-1}} + \frac{1}{Z_0 Z_{-1}} & \frac{\alpha}{Z_0} \\ \frac{1}{\alpha Z_0} & \frac{Z_{-1}}{Z_0} \end{pmatrix}$$

$$Z_{n+1} Z_{n-1} = 1 + Z_n^2, \qquad \{Z_n, Z_{n-1}\} = \gamma Z_n Z_{n-1}$$

$\mathrm{tr}\, M$ − integral of motion

Different variables

$$M = \begin{pmatrix} u+v & \alpha\sqrt{v/u} \\ \alpha^{-1}\sqrt{v/u} & u^{-1} \end{pmatrix}, \qquad \{u,v\} = 2\gamma uv$$

$$u_{n+1} = u_n + v_n$$

$$v_{n+1} = u_n^{-1} v_n (u_n + v_n)^{-1}$$

$(P,Q) \leftrightarrow (u,v)$ nontrivial canonical transformation

Easier in quantum variant

Quantization

$$uv = q^2 vu, \quad q = e^{i\gamma}$$

$$\gamma = \pi\tau, \quad \tau = \frac{\omega'}{\omega}, \quad \omega\omega' = -\frac{1}{4} \quad \text{half periods}$$

$$L_2(\mathbb{R}), \quad uf(x) = e^{-i\pi x/\omega} f(x), \quad vf(x) = f(x + 2\omega')$$

$$\tau > 0, \quad \omega, \omega' - \text{pure imaginary}$$

Domain: $f = e^{-x^2} e^{\beta x} P(x)$

u, v – positive, essentially selfadjoint

$$u_{n+1} = U^{-1} u_n U, \quad v_{n+1} = U^{-1} v_n U$$

$$E(u) f(x) = e^{i\pi x^2} f(x)$$

$$F f(x) = \int e^{-2\pi i x y} f(y)\, dy$$

$$uF = Fv, \quad vF = Fu^{-1}$$

$$\Phi(u) f(x) = \gamma(x) f(x)$$

$\gamma(x)$ – modular quantum dilogarithm

$$\gamma(x) = \exp -\frac{1}{4} \int \frac{e^{ixt}}{\sin \omega t \sin \omega' t} \frac{dt}{t}$$

$$\frac{\Phi(qu)}{\Phi(q^{-1}u)} = \frac{1}{1+u}$$

$$U = E^{-1}(u) \frac{1}{\Phi(v)} E^{-1}(u)$$

Return to Q, P

Begin with the diagonilization of $\operatorname{tr} M$

$$\operatorname{tr} M\psi = (u + u^{-1} + v)\psi(x,s) = (e^{i\pi s/\omega} + e^{-i\pi s/\omega})\psi$$

$$\psi(x,s) = \gamma(x+s-\omega'')\gamma(x-s-\omega'')e^{-i\pi(x-\omega'')^2}, \quad \omega'' = \omega+\omega'$$

$$\gamma(x-\omega'') \sim \frac{c}{x}, \quad \omega'' \to \omega'' - i0$$

Orthonormal (Kashaev, 2002)

$$\int \overline{\psi(x,s)}\psi(x,s')dx = \frac{1}{\rho(s)}\Big(\delta(s-s') + \delta(s+s')\Big)$$

$$\rho(s) = -4\sin\frac{\pi s}{\omega}\sin\frac{\pi s}{\omega'}$$

$$\int_0^\infty \psi(x,s)\overline{\psi(y,s)}\rho(s)ds = \delta(x-y)$$

$$R: \quad f(x) \to g(s) = \int_{-\infty}^{\infty} f(x)\psi(x,s)dx$$

$$g(s) = g(-s)$$

$$R: \quad L_2(\mathbb{R}) \to L_2(\mathbb{R}_+, \rho)$$

$$Z_0 R = q^{1/4} u^{1/2} v^{-1/2} R = Ri(r - r^{-1})(z - z^{-1})^{-1} = i\frac{\mathrm{Sh}\, Q}{\mathrm{Sh}\, P}$$

$$rf(s) = f(s + \omega'), \quad zf(s) = e^{-i\pi s/\omega} f(s), \quad zr = qrz$$

order of factors, even to even, essentially selfadjoint

$$\rho(s) = M(s)M(-s), \quad Mg(s) = M(s)g(s) = h(s)$$

$$M(s) = e^{2i\pi s(s-\omega'')}\gamma(2s + \omega''), \quad \overline{M(s)} = M(-s)$$

$$Z_0 RM = r^{-1} + \frac{1}{z - z^{-1}} r \frac{1}{z - z^{-1}} \text{ – selfadjoint in } L_2(\mathbb{R})$$

$$h(-s) = S(s)h(s), \quad S(s) = \frac{M(-s)}{M(s)}$$

$$\Pi = \frac{1}{2}\Big(\delta(s - s') + S(s)\delta(s + s')\Big)$$

Π – projector, $\Pi L_2(\mathbb{R})$

Evolution

$$RUR^{-1}h(s) = e^{-2\pi i s^2} h(s)$$

Conclusion:
The natural Hilbert space for the zero modes is subspace of $L_2(\mathbb{R})$ defined via reflection coefficient $S(s)$