NEW ACTION FOR THE HILBERT-EINSTEIN EQUATIONS

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Abstract. The Hilbert-Einstein equations are derived in the formalism employing the imbedding of the space-time into linear 10-dimensional space. An extra antisymmetric tensor field is needed for this task.

Einstein's Theory of Gravitation is the most ingenious achievement in the Theoretical Physics. It has dramatic history, experimental confirmations and beautiful geometric formulation. Its hamiltonian formulation (see e. g. [1] and refs. to fundamental papers of Dirac, ADM and others) allows for the formal quantization. However it is this aspect of the theory which is still considered unsatisfactory due to the perturbative nonrenormalizability. Thus the alternative proposals are periodically developed, most prominent of which are connected with the String Theory. In this note I discuss one more possibility to modify the mathematical formulation of the theory retaining the main Hilbert-Einstein equations.

First I shall remind the basics of the embedding approach to the description of metric on space-time, which was discussed by many people [2]–[4]. Then I make a trick by extending the set of degrees of freedom by promoting the one-forms to full covariant vector fields. As a result there will be no extra derivative in the new variational equation of motion, whih is the drawback of the Redge-Teitelboim equations.

I begin with the parametrization of the 4-dimensional space-time M^4 by imbedding into 10-dimensional Euclidian space $\mathbb{R}^{10}$

$$f^A = f^A(x^\mu),$$

where $f^A, A = 1, \dots 10$ and $x^\mu, \mu = 1, 2, 3, 4$ are coordinates in $\mathbb{R}^{10}$ and M^4, correspondingly. For simplicity I shall use the Euclidean signature on M^4, which can be easily changed into the Lorentzian one; so the metric on M^4 and Christoffel's symbols are given by

$$g_{\mu\nu} = \partial_\mu f^A \partial_\nu f^A$$

and

$$\Gamma_{\lambda,\mu\nu} = \partial_\lambda f^A \partial_\mu \partial_\nu f^A$$

(see e. g. V. Fock monograph [5]). Let us use these formulas to express the curvature tensor. For this I employ the formula

$$R_{\mu\nu,\alpha\beta} = \frac{1}{2}\big(\partial_\mu\partial_\beta g_{\nu\alpha} + \partial_\nu\partial_\alpha g_{\mu\beta} - \partial_\mu\partial_\alpha g_{\nu\beta} - \partial_\nu\partial_\beta g_{\mu\alpha} - \\ - g^{\lambda\sigma}(\Gamma_{\lambda,\mu\alpha}\Gamma_{\sigma,\nu\beta} - \Gamma_{\lambda,\mu\beta}\Gamma_{\sigma,\nu\alpha})\big),$$

also presented in [5].

Substitution the expressions for $g_{\mu\nu}$ and $\Gamma_{\lambda,\mu\nu}$ via derivatives of f^A gives

$$R_{\mu\nu,\alpha\beta} = \Pi^{AB}(f^A_{\mu\alpha}f^B_{\nu\beta} - f^A_{\nu\alpha}f^B_{\mu\beta}).$$

Here I use the short notations

$$f^A_{\mu\alpha} = \partial_\mu\partial_\alpha f^A, \quad f^A_\mu = \partial_\mu f^A$$

and by Π^{AB} denote the projector

$$\Pi^{AB} = \delta^{AB} - g^{\lambda\sigma}f^A_\lambda f^B_\sigma$$

on the subspace, orthogonal to the tangent space to M^4 at point x^μ.

$$\Pi^{AB}f^B_\mu = 0, \quad \Pi^{AB}f^{\mu,B} = 0,$$

where $f^{\mu,A} = g^{\mu\nu}f^A_\nu$.

The expression for $R_{\mu\nu,\alpha\beta}$ clearly is compatible with the symmetry properties of the curvature tensor. Let us stress two features of this formula.

1. Naively we could expect, that $R_{\mu\nu,\alpha\beta}$ is linear in the third derivatives of f^A, being linear in the second derivatives of $g_{\mu\nu}$. However the third derivatives cancel and $R_{\mu\nu,\alpha\beta}$ is a quadratic form of the second derivatives of f^A.

2. Expression for $R_{\mu\nu,\alpha\beta}$ is covariant in spite of the fact, that it contains only ordinary derivatives. Indeed the infinitesimal coordinate variation

$$\delta f^A = -\epsilon^\lambda\partial_\lambda f^A$$

induces transformations

$$\delta f^A_\mu = -\partial_\mu\epsilon^\lambda\partial_\lambda f^A - \epsilon^\lambda\partial_\lambda f^A_\mu,$$

corresponding to that of the covariant vector field and

$$\delta f^A_{\mu\alpha} = -\partial_\mu\epsilon^\lambda f^A_{\lambda\alpha} - \partial_\alpha\epsilon^\lambda f^A_{\mu\lambda} - \epsilon^\lambda\partial_\lambda f^A_{\mu\alpha} - \partial_\mu\partial_\alpha\epsilon^\lambda f^A_\lambda.$$

The first three terms here correspond to the transformation of the covariant tensor field and so are satisfactory. The last unwanted term is proportional to the linear combination of vectors f^A_λ and is annihilated by projector Π^{AB}.

The contracted tensor $R_{\mu\alpha} = g^{\nu\beta} R_{\mu\nu,\alpha\beta}$ and scalar curvature R can be written via derivatives of the contravariant vector field

$$f^{\mu A} = g^{\mu\nu} f^A_\nu$$

as follows

$$\begin{aligned} R_{\mu\alpha} &= \Pi^{AB}(\partial_\alpha f^A_\mu \partial_\beta f^{\beta B} - \partial_\beta f^A_\mu \partial_\alpha f^{\beta B}) \\ R &= \Pi^{AB}(\partial_\alpha f^{\alpha A} \partial_\beta f^{\beta B} - \partial_\beta f^{\alpha A} \partial_\alpha f^{\beta B}) \end{aligned}$$

Indeed

$$\partial_\alpha f^{\mu A} = g^{\mu\nu} \partial_\alpha f^A_\nu + \partial_\alpha g^{\mu\nu} f^A_\nu$$

and the second term is annihilated by projector.

With these formulas we prepared our main trick. Let us take f^A_μ as generic covariant vector field, put

$$g_{\mu\nu} = f^A_\mu f^A_\nu, \quad f^{\mu A} = g^{\mu\nu} f^A_\nu, \quad g^{\mu\nu} = f^{\mu A} f^{\nu A}$$

and take

$$\mathcal{L} = \sqrt{g} R$$

as lagrangian. It is quadratic in the first derivatives of contravariant vector field $f^{\mu A}$. Then we calculate the variational equations and supplement them by extra equations of motion

$$\partial_\mu f^A_\nu - \partial_\nu f^A_\mu = 0$$

with solution

$$f^A_\mu = \partial_\mu f^A,$$

which will be produced by an additional lagrangian of BF type

$$\mathcal{L}_1 = B^{\mu\nu,A}(\partial_\mu f^A_\nu - \partial_\nu f^A_\mu),$$

where $B^{\mu\nu,A}$ is a set of antisymmetric contravariant tensor densities. Superficially this trick has nothing to do with Hilbert-Einstein equations. However to my own surprise the variation of $\mathcal{L}$ will contain $R_{\alpha\mu}$. More exactly we have formula

$$\delta \int \mathcal{L} d^4x = \delta f^{\alpha A} \Gamma^A_\alpha,$$

where

$$\begin{aligned} \Gamma^A_\alpha = 2\sqrt{g}\big(-\Pi^{AB} f^{\mu,C} + \Pi^{BC} f^{\mu,A} + \Pi^{AC} f^{\mu,B}\big) T^{CB}_{\mu\alpha} - \\ - \sqrt{g} g_{\alpha\mu}\big(\Pi^{AB} f^{\mu,C} + \Pi^{BC} f^{\mu,A} + \Pi^{AC} f^{\mu,B}\big) T^{BC}. \end{aligned}$$

Here $T^{AB}_{\mu\alpha}$ and T^{AB} are quadratic forms of the first derivatives, entering the expressions for $R_{\mu\alpha}$ and R

$$T^{AB}_{\mu\alpha} = \partial_\alpha f^A_\mu \partial_\beta f^{\beta B} - \partial_\beta f^A_\mu \partial_\alpha f^{\beta B},$$
$$T^{AB} = \partial_\alpha f^{\alpha A} \partial_\beta f^{\beta B} - \partial_\beta f^{\alpha A} \partial_\alpha f^{\beta B}.$$

Furthermore, the variation of $\int \mathcal{L}_1 d^4x$ gives expression $\delta f^{\alpha A}\Sigma^A_\alpha$, where

$$\Sigma^A_\alpha = \Pi^{AB}\partial_\mu B^{\mu\nu,B} g_{\alpha\nu} - f^A_\nu f^B_\alpha \partial_\mu B^{\mu\nu,B}.$$

Altogether the full set of equations of motion is

$$\begin{cases} \Gamma^A_\alpha + \Sigma^A_\alpha = 0, \\ \partial_\mu f^A_\nu - \partial_\nu f^A_\mu = 0 \end{cases}.$$

Projecting the first line on f^A_μ we get equations

$$R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R + T_{\alpha\mu} = 0$$
$$T_{\alpha\mu} = \frac{1}{\sqrt{g}} g_{\mu\nu} \partial_\sigma B^{\sigma\nu,B} f^B_\alpha,$$

which together with the second line is equivalent to Hilbert-Einstein equations with energy-momentum tensor produced by B field.

S. A. Paston [6] has shown that the vertical contribution to Γ^A_α vanishes after the second set of equations of motion is taken into account. Thus we have besides the Hilbert-Einstein equations the condition

$$\Pi^{AB}\partial_\mu B^{\mu\nu,B} = 0$$

and so only the horisontal part of $\partial_\mu B^{\mu\nu,B}$ remains unfixed.

Introduction of the B-field is the price for the success of my trick. One must return to this problem of interpretation after more close inspection of all equations.

We finish this note by deriving the variational equations. Due to the form of R it is natural to make the variations via contravariant vector field $f^{\mu,A}$. However the projector Π^{AB} contains also covariant fields

$$\Pi^{AB} = \delta^{AB} - f^A_\mu f^{\mu B}.$$

So the first thing is to find its variation. Using

$$f^A_\mu = g_{\mu\nu} f^{\nu A},$$

we get

$$\begin{aligned}\delta f_\mu^A &= \delta g_{\mu\nu} f^{\nu A} + g_{\mu\nu}\delta f^{\nu A} = \\ &= -g_{\mu\sigma}\delta g^{\sigma\rho} g_{\rho\nu} f^{\nu A} + g_{\mu\nu}\delta f^{\nu A} = \\ &= -g_{\mu\sigma}(\delta f^{\sigma C} f^{\rho C} + f^{\sigma C}\delta f^{\rho C}) g_{\rho\nu} f^{\nu A} + g_{\mu\nu}\delta f^{\nu A} = \\ &= \delta f^{\sigma C}(g_{\mu\sigma}\delta^{AC} - g_{\mu\sigma} f^{\rho C} f_\rho^A - f_\mu^C f_\sigma^A) = \\ &= \delta f^{\sigma C}(g_{\mu\sigma}\Pi^{AC} - f_\mu^C f_\sigma^A),\end{aligned}$$

from which follows an elegant formula

$$\delta\Pi^{AB} = -\delta f_\mu^A f^{\mu B} - f_\mu^A \delta f^{\mu B} = -\delta f^{\sigma C}(\Pi^{AC} f_\sigma^B + \Pi^{CB} f_\sigma^A).$$

The derivative of Π^{AB} we prefer to express via the derivative of the covariant field. The similar calculation gives

$$\partial_\alpha \Pi^{AB} = -\partial_\alpha f_\sigma^C(\Pi^{AC} f^{\sigma B} + \Pi^{CB} f^{\sigma A}).$$

Finally for the variation and derivative of $\sqrt{g}$ we have

$$\delta\sqrt{g} = -\sqrt{g}\delta f^{\sigma A} f_\sigma^A, \quad \partial_\alpha\sqrt{g} = \sqrt{g} f^{\sigma A}\partial_\alpha f_\sigma^A.$$

Now we have

$$\begin{aligned}\Gamma_\alpha^A = -2\partial_\alpha(\sqrt{g}\Pi^{AB}\partial_\beta f^{\beta B}) + 2\partial_\beta(\sqrt{g}\Pi^{AB}\partial_\alpha f^{\beta B}) - \\ - \sqrt{g}(\Pi^{AB} f_\alpha^C + \Pi^{AC} f_\alpha^B) T^{BC} - \sqrt{g} f_\alpha^A R\end{aligned}$$

and after differentiating we get the main formula for Γ_α^A. It is instructive to observe that the second derivatives cancel.

The variation of $\mathcal{L}_1$ gives after using the expression of δf_μ^A via $\delta f^{\mu,A}$

$$\delta\mathcal{L}_1 = -2\partial_\mu B^{\mu\nu,A}\delta f_\nu^A = -2\partial_\mu B^{\mu\nu,C}(\Pi^{AC} g_{\nu\alpha} - f_\nu^A f_\alpha^C)\delta f^{\alpha,A},$$

so that

$$\Sigma_\alpha^A = -2\Pi^{AC} g_{\nu\alpha}\partial_\mu B^{\mu\nu,C} + 2\partial_\mu B^{\mu\nu,C} f_\alpha^C f_\nu^A$$

and again we have explicit separation along tangent vectors and orthogonal subspace.

With this derivation I finish this note. Lot of work for the interpretation of the presented trick is ahead. The first thing to do is to compare the hamiltonian formulation, following from our lagrangian with that of Dirac and ADM.

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