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EVALUATION OF AN INFINITE PRODUCT OF SPECIAL MATRICES

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An important role in studying the integrable models of Field Theory is played by matrix-functions of complex variable of a special form [1]. The simplest example is provided by a rational matrix:

$$L_0(z) = \frac{z+B}{z+\mu}, \tag{1}$$

where B is a matrix of size $n \times n$ and μ is a complex number. It is natural to call it the matrix Weierstrass factor for the complex plane $\mathbb{C}$ (i.e., a meromorphic function on $\mathbb{C}$ with one pole and such that $L(\infty) = 1$).

The next interesting example is given by a matrix Weierstrass factor for a strip. This function L_1 is meromorphic in the strip $\{ z \in \mathbb{C} : 0 < \operatorname{Re} z \leqslant 1 \}$ has only one pole in it and is regular at infinity, i.e.,

$$L_1(z) \to \mathcal{D}_{\pm}, \qquad \operatorname{Im} z \to \pm\infty,$$

where $\mathcal{D}_{\pm}$ are non-degenerate diagonal matrices. The boundary values of $L_1(z)$ satisfy the following relation:

$$L_1(i\lambda + 1) = A L_1(i\lambda) A^{-1}, \qquad \lambda \in \mathbb{R}, \tag{2}$$

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where $A = \mathrm{diag}(1, \varepsilon, \dots, \varepsilon^{n-1})$, $\varepsilon = \exp(\frac{2\pi i}{n})$. One can represent such a matrix-function as an infinite product of functions (1). For this purpose introduce the family of matrices

$$L^m(z) = A^m L_0(z+m) A^{-m} \tag{3}$$

and their finite product

$$L_1^N(z) = L^N(z) L^{N-1}(z) \dots L^{-N+1}(z) L^{-N}(z). \tag{4}$$

It is easy to show that the regularized limit

$$L_1(z) = \lim_{N\to\infty}{}' \, L_1^N(z) \tag{5}$$

satisfies (2).

For $n = 1$ formulae (3)-(5) are nothing but Euler's formulae for $\sin u$, so that

$$L_1(z) = \frac{\sin \pi(z+B)}{\sin \pi(z+\mu)}.$$

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We calculated $L_1(z)$ for $n = 2$ in [2]. In this case $A = \mathrm{diag}(1, -1)$ and

$$B = \begin{pmatrix} S_3 & S_- \\ S_+ & -S_3 \end{pmatrix}$$

so that $\mathrm{trace}(B) = 0$.

The limit in (5) is defined as follows

$$\lim_{N\to\infty}{}' \, L_1^N(z) = \lim_{N\to\infty} \left(A^{-N} D_N L_1^N(z) D_{-N} A^N \right),$$

where

$$D_N = \begin{pmatrix} N^{S_3} & 0 \\ 0 & N^{-S_3} \end{pmatrix}.$$

The limit matrix $L_1(z)$ has a form

$$L_1(z) = W^{-1} \tilde{L}_1(z) W,$$

$$W = \begin{pmatrix} h(S_3)^{-1} & 0 \\ 0 & h(S_3) \end{pmatrix}, \qquad h(z) = \sqrt{\frac{\Gamma(1+R-z)\Gamma(1-R-z)}{\pi^2(z^2-R^2)}}\, e^{-z},$$

$$\tilde{L}_1(z) = \frac{1}{\sin \pi(z+\mu)} \begin{pmatrix} \sin \pi(z+S_3), & S_- \sqrt{\dfrac{\sin^2 \pi S_3 - \sin^2 \pi R}{\pi^2(S_3^2 - R^2)}} \\ S_+ \sqrt{\dfrac{\sin^2 \pi S_3 - \sin^2 \pi R}{\pi^2(S_3^2 - R^2)}} & \sin \pi(z - S_3) \end{pmatrix},$$

where $R^2 = S_3^2 + S_+ S_-$.

We pose as a PROBLEM *the explicit calculation of the limit in* (5) *for every* n *in terms of known special functions.*

REFERENCES

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