3+1 DECOMPOSITION IN THE NEW ACTION FOR THE EINSTEIN THEORY OF GRAVITATION

L. D. FADDEEV

Abstract. The action of recently proposed formulation of Einstein Theory of Gravitation is written according to 3+1 decomposition of the space-time variables. The result coincides with known formulation of Dirac and Arnowitt-Deser-Misner.

Recently I proposed to use new dynamical variables to describe the gravitational field [1], [2]. In [2] the new formulation was shown to be equivalent to the classical one of Hilbert-Einstein. So the question arises why to do such an effort. The only answer I can give now is that I follow an old advice of Feynman — to generalize a theory one must work it out in many guises. And there is no doubt that we need to develop Einstein's theory further.

In this note I continue the work in [1], [2] and develop 3+1 decomposition of the space-time variables in the action functional. I shall show how traditional formulas of Dirac [3] and Arnowitt-Deser-Misner [4] appear in my formulation.

The set of dynamical variables introduced in [1], [2] consists of 40 components — 10 covariant vector fields $f^A_\mu(x)$ on four dimensional space-time M_4 with coordinates x^μ; thus $A = 1, \dots, 10$, $\mu = 0, 1, 2, 3$.

In terms of these variables I define metric

$$g_{\mu\nu} = f^A_\mu f^A_\nu$$

and linear connection

$$\Omega^\beta_{\alpha\mu} = f^{\beta A} \partial_\mu f^A_\alpha .$$

Here $f^{\mu A}(x)$ are contravariant vector fields

$$f^{\mu A} = g^{\mu\nu} f^A_\nu ,$$

where $g^{\mu\nu}$ is, as usual, inverse to $g_{\mu\nu}$

$$g_{\mu\sigma} g^{\sigma\nu} = \delta^\nu_\mu .$$

As in [1], [2] I do not bother with the subtleties of the pseudoriemannian signature, so all scalar products are euclidean. In such situation the separation of time and space variables x^0 and x^k seems somewhat artificial, but I continue to follow this convention to avoid minus signs. So I use a term "3+1 decomposition" instead of the space-time one.

The action is written more transparently via the contravariant components

$$A = \int \sqrt{g} S d^4 x,$$

where

$$S = \Pi^{AB}(\partial_\mu f^{\mu A} \partial_\nu f^{\nu B} - \partial_\mu f^{\nu A} \partial_\nu f^{\mu B})$$

and Π^{AB} is the "vertical" projector

$$\Pi^{AB} = \delta^{AB} - g_{\mu\nu} f^{\mu A} f^{\nu B}.$$

The expression S defines the scalar curvature of the connection $\Omega^{\beta}_{\alpha\mu}$. The full curvature tensor

$$S_{\alpha\beta,\mu\nu} = g_{\beta\gamma}\big(\partial_\mu \Omega^{\gamma}_{\alpha\nu} - \partial_\nu \Omega^{\gamma}_{\alpha\mu} + \Omega^{\sigma}_{\alpha\mu}\Omega^{\gamma}_{\sigma\nu} - \Omega^{\sigma}_{\alpha\nu}\Omega^{\gamma}_{\sigma\mu}\big)$$

is beautiffully expressed as

$$S_{\alpha\beta,\mu\nu} = \Pi^{AB}(\partial_\mu f^A_\alpha \partial_\nu f^B_\beta - \partial_\mu f^A_\beta \partial_\nu f^B_\alpha)$$

and

$$S = g^{\mu\alpha} g^{\nu\beta} S_{\alpha\beta,\mu\nu}.$$

All these formulas include only usual partial derivatives, but they are fully covariant with respect to the general coordinate transformations

$$\delta f^A_\mu = -\partial_\mu \epsilon^\nu f^A_\nu - \epsilon^\nu \partial_\nu f^A_\mu,$$

where ϵ^ν is a vector field, defining infinitesimal coordinate transformation.

In this note I shall explicitly realize the 3+1 decomposition of these formulas in coordinates $x^\mu = (x^0, x^k), k = 1,2,3$ and refer to $x^0 = t$ as time and to x^k as space variables. The main goal is to rewrite the action in hamiltonian-like form.

The first observation is that S contains time derivatives only linearly. This allows to develop the reduction formalism following the general ideas of my paper with R. Jackiw [5]. There the dynamical variables entering the original lagrangian are divided into three classes: canonical, excludable and Lagrange multipliers. To exclude the variables of second class one is allowed to use equations of motion which do not contain the time derivatives.

Among the equations of motion, which are derived in [2], there is a set of equations which express the vanishing of the torsion of the connection $\Omega^{\beta}_{\alpha\mu}$

$$\Omega^{\beta}_{\alpha\mu} = \Omega^{\beta}_{\mu\alpha}.$$

Out of these 24 equations 12 do not contain the time derivatives

$$\Omega^0_{ik} = \Omega^0_{ki}, \quad \Omega^l_{ik} = \Omega^l_{ki}$$

and I shall use them in the reduction of action in what follows.

The formulas I plan to derive should be covariant with respect to coordinate transformation, generated by vector fields $\epsilon^i(x)$, obtained from $\epsilon^\mu(x)$ by restriction

$$\epsilon^0(x) = 0, \quad \partial_t \epsilon^i(x) = 0.$$

The covariant vector fields f^A_k are compatible with this requirement

$$\delta f^A_k = -\partial_k \epsilon^l(x) f^A_l - \epsilon^l \partial_l f^A_k.$$

Furthermore the component f^{0A} defines scalars

$$\delta f^{0A} = -\epsilon^k \partial_k f^{0A}.$$

I begin by writing

$$\sqrt{g} S = \Sigma + \mathcal{H},$$

where Σ contains all terms with time derivatives

$$\Sigma = 2\sqrt{g}\Pi^{AB} g^{\mu l}(\partial_l f^A_\mu \partial_t f^{0B} - \partial_l f^{0A} \partial_t f^B_\mu)$$

and

$$\mathcal{H} = \sqrt{g} g^{k\sigma} g^{l\rho}(\partial_k f^A_\sigma \partial_l f^B_\rho - \partial_l f^A_\sigma \partial_k f^B_\rho).$$

We can interpret Σ as one form using substitution

$$\partial_t f^A_\mu \to df^A_\mu.$$

In this guise the action A is an explicit example of general scheme in [5].

Now we proceed to realize the promised separation. The covariant 3-dimensional metric γ_{ik} is given by

$$\gamma_{ik} = f^A_i f^A_k$$

and the components of 4-dimensional contravariant metric $g^{\mu k}$, which we need, can be expressed via γ^{ik}, g^{0i}, g^{00}, which are 3-minesional tensor, vector and scalar, correspondingly,

$$g^{ik} = \gamma^{ik} + \frac{g^{0i} g^{0k}}{g^{00}}.$$

The 4-dimensional determinant can be written as

$$g = \gamma / g^{00},$$

where γ is determinant of metric γ_{ik}. We shall also see, that terms containing f^A_0 will always have the form

$$g^{00} f^A_0 + g^{0k} f^A_k = f^{0A}.$$

Let us begin our rearrangement with one-form Σ. We have

$$\Sigma = 2\sqrt{g}\Pi^{AB}\big[(\gamma^{ml} + \frac{g^{0m}g^{0l}}{g^{00}})(\partial_l f_m^A \partial_t f^{0B} - \partial_l f^{0A}\partial_t f_m^B) \\ + g^{0l}(\partial_l f_0^A \partial_t f^{0B} - \partial_l f^{0A}\partial_t f_0^B)\big]$$

and immediately see, that terms, proportional to g^{0l} contain combinations

$$\frac{g^{0m}}{g^{00}}\partial_l f_m^A + \partial_l f_0^A = \frac{1}{g^{00}}\partial_l f^{0A} - \frac{1}{g^{00}}(\partial_l g^{0m} f_m^A + \partial_l g^{00} f_0^A)$$
$$\frac{g^{0m}}{g^{00}}\partial_t f_m^A + \partial_t f_0^A = \frac{1}{g^{00}}\partial_t f^{0A} - \frac{1}{g^{00}}(\partial_t g^{0m} f_m^A + \partial_t g^{00} f_0^A)$$

and the second terms in the RHS are annihilated by vertical projector Π^{AB}. After this observation we see, that these terms cancel and we get the satisfactory expression

$$\Sigma = 2\sqrt{g}\Pi^{AB}\gamma^{kl}(\partial_l f_k^A \partial_t f^{0B} - \partial_l f^{0A}\partial_t f_k^B).$$

Let us do the same for $\mathcal{H}$ and separate the contributions, corresponding to $(\sigma,\rho) = (0,0)$, $(m,0)$, $(0,n)$ and (m,n) . The $(0,0)$ component vanishes due to antisymmetry. The $(m,0)$ and $(0,n)$ components coincide after change of mute indeces and give

$$Q_1 = 2\sqrt{g}\Pi^{AB}\gamma^{ln}g^{k0}(\partial_k f_0^A \partial_l f_n^B - \partial_l f_0^A \partial_k f_n^B).$$

The (m,n) contribution gives

$$Q_2 = \sqrt{g}\Pi^{AB}g^{km}g^{ln}S^{AB}_{kl,mn},$$

where

$$S^{AB}_{kl,mn} = \partial_k f_m^A \partial_l f_n^B - \partial_k f_n^A \partial_l f_m^B,$$

and substituting g^{km} via γ^{km}, g^{0k}, g^{k0} we get $Q_2 = Q_3 + Q_4$, where

$$Q_3 = \sqrt{g}\Pi^{AB}\gamma^{km}\gamma^{ln}S^{AB}_{kl,mn}$$

and

$$Q_4 = 2\sqrt{g}\frac{g^{0k}}{g^{00}}\gamma^{ln}g^{0m}S^{AB}_{kl,mn}.$$

Combining Q_1 and Q_4 and using the same trick as before we get

$$Q_1 + Q_4 = 2\sqrt{g}\frac{g^{0k}}{g^{00}}\Pi^{AB}\gamma^{ln}(\partial_k f^{0A}\partial_l f_n^B - \partial_l f^{0A}\partial_k f_n^B).$$

Thus we get

$$\mathcal{H} = T_0 + T_1$$
$$T_1 = Q_1 + Q_4, \quad T_0 = Q_3$$

and their expressions are satisfactory also.

Now it is time to reduce the vertical projector Π^{AB}. We have

$$\Pi^{AB} = \delta^{AB} - g^{\mu\nu} f^A_\mu f^B_\nu =$$
$$= \delta^{AB} - g^{ik} f^A_i f^B_k - -g^{i0} f^A_i f^B_0 - g^{0k} f^A_0 f^B_k - g^{00} f^A_0 f^B_0 =$$
$$= \delta^{AB} - \gamma^{ik} f^A_i f^B_k - \Big(\frac{g^{0i} g^{0k}}{g^{00}} f^A_i f^B_k + g^{0k} f^A_0 f^B_k +$$
$$+ g^{i0} f^A_i f^B_0 + f^A_0 f^B_0 g^{00}\Big).$$

The first two terms define the 3-dimensional vertical projector and the last can be rewritten as

$$\frac{1}{g^{00}} (g^{0i} f^A_i + g^{00} f^A_0)(g^{0k} f^B_k + g^{00} f^B_0) = \frac{1}{g^{00}} f^{0A} f^{0B}.$$

Thus we have

$$\Pi^{AB} = \delta^{AB} - \gamma^{ik} f^A_i f^B_k - \frac{1}{g^{00}} f^{0A} f^{0B}.$$

Let us mention, that the last term has proper normalization because

$$f^{0A} f^{0A} = g^{00}.$$

Now I substitute this expression for Π^{AB} into Σ, $\mathcal{H}_0$ and $\mathcal{H}_k$.

Begin with Σ: we get three contributions according to the form of Π^{AB}. The first is

$$\Sigma_1 = 2\sqrt{g}\gamma^{kl} (\partial_l f^A_k \partial_t f^{0A} - \partial_l f^{0A} \partial_t f^A_k).$$

The last factor can be rewritten as

$$\partial_t (\partial_l f^A_k f^{0A}) - \partial_l (\partial_t f^A_k f^{0A}) = \partial_t \Omega^0_{kl} - \partial_l \Lambda_k,$$

where I remind the notation for $\Omega^\beta_{\alpha\mu}$ and denote

$$\Lambda_k = \partial_t f^A_k f^{0A}.$$

Thus we have

$$\Sigma_1 = 2\sqrt{g}\gamma^{kl} (\partial_t \Omega^0_{kl} - \partial_l \Lambda_k).$$

The first term here is quite satisfactory, it is almost of Darboux form.

Now consider the second term

$$\Sigma_2 = -2\sqrt{g}\gamma^{kl}\gamma^{mn} \big[(f^A_m \partial_l f^A_k)(f^B_n \partial_t f^{0B}) - (f^A_m \partial_l f^{0A})(f^B_n \partial_t f^B_k)\big] =$$
$$= 2\sqrt{g}\gamma^{kl}\gamma^{mn} \big(\omega_{m.kl} \Lambda_n - (f^B_n \partial_t f^B_k)\Omega^0_{ml}\big).$$

Here I used the orthonormality of f^A_k and f^{0A} to rewrite

$$f^B_n \partial_t f^{0B} = -\partial_t f^B_n f^{0B} = -\Lambda_n$$

and

$$f^A_m \partial_l f^{0A} = -\partial_l f^A_m f^{0A} = -\Omega^0_{ml}$$

and introduce the 3-dimensional connection

$$\omega_{m,kl} = f^A_m \partial_l f^A_k.$$

Finally the third contribution is given by

$$\Sigma_3 = -\sqrt{g}\gamma^{kl}\frac{1}{g^{00}}[\Omega^0_{kl}\partial_t g^{00} - \partial_l g^{00}\Lambda_k],$$

where I used that

$$f^{0A}\partial_t f^{0A} = \frac{1}{2}\partial_t g^{00}, \quad f^{0A}\partial_l f^{0A} = \frac{1}{2}\partial_l g^{00}.$$

Let us collect all contrbutions containing Λ_k

$$\Lambda = 2\sqrt{g}\gamma^{kl}\big(-\partial_l\Lambda_k + \omega^m_{kl}\Lambda_m + \frac{1}{2}\frac{\partial_l g^{00}}{g^{00}}\Lambda_k\big)$$

and compare it with

$$\partial_l(\sqrt{g}\gamma^{kl}\Lambda_k) = \partial_l\big(\frac{\sqrt{\gamma}}{\sqrt{g^{00}}}\gamma^{kl}\Lambda_k\big) =$$

$$= \sqrt{g}\big(\omega^m_{ml} - \frac{1}{2}\frac{\partial_l g^{00}}{g^{00}}\big)\gamma^{kl}\Lambda_k + \sqrt{g}\partial_l\gamma^{kl}\Lambda_k + \sqrt{g}\gamma^{kl}\partial_l\Lambda_k.$$

Using the vanishing of covarinat derivative of γ^{kl}

$$\partial_l\gamma^{kl} + \omega^k_{ml}\gamma^{ml} + \omega^l_{ml}\gamma^{mk} = 0$$

we get

$$\Lambda = -2\partial_l(\sqrt{g}\gamma^{kl}\Lambda_k) + 2\sqrt{g}\gamma^{kl}\Lambda_k(\omega^m_{ml} - \omega^m_{lm}).$$

The second term in the RHS disappears due to mentioned above time-independent equations of motion. Indeed we have

$$\Omega^m_{ik} = f^{mA}\partial_k f^A_i = g^{m\sigma}f^A_\sigma\partial_k f^A_i =$$

$$= \big(\gamma^{mn} + \frac{g^{m0}g^{n0}}{g^{00}}\big)f^A_n\partial_k f^A_i + g^{m0}f^A_0 f^A_i = \omega^m_{ik} + \frac{g^{m0}}{g^{00}}\Omega^0_{ik}$$

and ω^m_{ik} is symmetric to interchange $i \leftrightarrow k$ together with Ω^m_{ik} and Ω^0_{ik}.

Thus the full contribution containing Λ_k is a pure divergence and can be omitted.

Consider now the expression

$$\gamma^{kl}\gamma^{mn}\Omega^0_{ml}(f^A_n\partial_t f^A_k).$$

Due to symmetry of Ω^0_{ml} it can be rewritten as

$$\frac{1}{2}\gamma^{kl}\gamma^{mn}(f^A_n\partial_t f^A_k + f^A_k\partial_t f^A_n)\Omega^0_{ml} = -\frac{1}{2}\partial_t\gamma^{ml}\Omega^0_{ml}.$$

Using this we have

$$\Sigma = \sqrt{g}\big(2\gamma^{kl}\partial_t\Omega_{kl} + \partial_t\gamma^{kl}\Omega^0_{kl} - \gamma^{kl}\Omega_{kl}\frac{\partial_t g^{00}}{g^{00}}\big) = \\ = q^{kl}\partial_t\Pi_{kl} + \partial_t(\sqrt{g}\gamma^{kl}\Omega_{kl}),$$

where

$$q^{kl} = \gamma\gamma^{kl}, \quad \Pi_{kl} = \frac{1}{\sqrt{\gamma g^{00}}}\Omega^0_{kl}.$$

The second term can be dropped and we obtain the canonical expression for the one form Σ. The normalization of canonical pairs — q^{ik} as contravariant density of weight 1 and Π_{ik} as covariant density of weight $-1/2$ — appeared first in the paper of Schwinger [6]. I used it in the survey [7].

Let us turn now to $\mathcal{H}$. First take T_1

$$T_1 = 2\sqrt{g}\frac{g^{k0}}{g^{00}}\Pi^{AB}\gamma^{lm}(\partial_k f^{0A}\partial_l f^B_m - \partial_l f^{0A}\partial_k f^B_m)$$

and consider three contrbutions according to form of Π^{AB}.

In the first we use

$$\partial_k f^{0A}\partial_l f^A_m - \partial_l f^{0A}\partial_k f^A_m = \partial_k(f^{0A}\partial_l f^A_m) - \partial_l(f^{0A}\partial_k f^A_m) = \\ = \partial_k\Omega^0_{ml} - \partial_l\Omega^0_{mk}.$$

In the second we get

$$-\gamma^{pq}\big[(f^A_p\partial_k f^{0A})(f^B_q\partial_l f^B_m) - (f^A_p\partial_l f^{0A})(f^B_q\partial_k f^B_m)\big] = \\ = \omega^p_{ml}\Omega^0_{pk} - \omega^p_{mk}\Omega^0_{pl}.$$

Finally in the third term we have

$$-\frac{1}{g^{00}}\big[(f^{0A}\partial_k f^{0A})(f^{0B}\partial_l f^B_m) - (f^{0A}\partial_l f^{0A})(f^{0B}\partial_k f^B_m)\big] = \\ = -\frac{1}{2}\frac{\partial_k g^{00}}{g^{00}}\Omega^0_{ml} + \frac{1}{2}\frac{\partial_l g^{00}}{g^{00}}\Omega^0_{mk}.$$

Combining all together we get

$$T_1 = 2\frac{g^{k0}}{g^{00}}\sqrt{g}\gamma^{lm}\big[\partial_k\Omega^0_{ml} - \partial_l\Omega^0_{mk} - \frac{1}{2}\frac{\partial_k g^{00}}{g^{00}}\Omega^0_{ml} + \\ + \frac{1}{2}\frac{\partial_l g^{00}}{g^{00}}\Omega^0_{mk} + \omega^p_{ml}\Omega^0_{pk} + \omega^p_{mn}\Omega^0_{pl}\big].$$

Taking into account symmetry of ω^p_{ml} and Ω^0_{pl} this can be written as

$$T_1 = \lambda^k\mathcal{H}_k,$$

where we introduce Lagrange multipliers

$$\lambda^k = \frac{g^{0k}}{g^{00}}$$

and

$$\mathcal{H}_k = 2\big[\nabla_k(q^{ml}\Pi_{ml}) - \nabla_l(q^{ml}\Pi_{mk})\big].$$

Finally consider T_3 and take into account, that the first two terms in Π^{AB} give 3-dimensional analogue of the vertical projector. Its contribution to T_0 is

$$\sqrt{g}\gamma^{km}\gamma^{ln}\Pi^{AB(3)}S^{AB}_{kl,mn} = \sqrt{g}S^{(3)},$$

where $S^{(3)}$ is scalar curvature of metric γ_{ik} and connection ω^m_{nl}. The last term in Π^{AB} gives

$$-\frac{f^{0A}f^{0B}}{g^{00}}S^{AB}_{kl,mn} = -\frac{1}{g^{00}}(\Omega^0_{mk}\Omega^0_{nl} - \Omega^0_{nk}\Omega^0_{ml}).$$

With this T_0 can be rewritten as

$$T_0 = \lambda_0\mathcal{H}_0,$$

where Lagrange multiplier λ_0 is given by

$$\lambda_0 = \frac{\sqrt{g}}{\gamma} = \frac{1}{\sqrt{g^{00}}\gamma} = \frac{1}{\sqrt{g}g^{00}}$$

and

$$\mathcal{H}_0 = \gamma S^{(3)} - q^{km}q^{ln}(\Pi_{km}\Pi_{ln} - \Pi_{lm}\Pi_{kn}).$$

This finishes calculations in this note. Let us remind that in it we used the change of 40 variables f^A_μ to the set $(f^A_k, f^{0A}, g^{00}, g^{0k})$. Superficially we have here 44 components, however we have 4 constraints

$$f^A_k f^{0A} = 0, \quad f^{0A}f^{0A} = g^{00}.$$

The main result is the formula for the action in 3+1 decomposition

$$A = \int d^3x \int dt(q^{ik}\partial_t\Pi_{ik} + \lambda^0\mathcal{H}_0 + \lambda^k\mathcal{H}_k),$$

which coincides with formulas of Dirac and ADM. Thus it is shown once more, that my proposal is equivalent to the classical formalism of Hilbert-Einstein.

However, as I already said in the beginning, this formulation could be a point of departure for the generalization not evident in the classical formulation.

References

[1] L. D. Faddeev, "New action for the Hilbert-Einstein equations," arXiv:0906.4639 [hep-th].
[2] L. D. Faddeev, "New variables for the Einstein theory of gravitation," arXiv:0911.0282 [hep-th].
[3] P. A. M. Dirac, "The Theory of gravitation in Hamiltonian form," Proc. Roy. Soc. Lond. A **246** (1958) 333.
[4] R. L. Arnowitt, S. Deser and C. W. Misner, "Canonical variables for general relativity," Phys. Rev. **117** (1960) 1595.
[5] L. D. Faddeev and R. Jackiw, "Hamiltonian Reduction of Unconstrained and Constrained Systems," Phys. Rev. Lett. **60** (1988) 1692.
[6] J. S. Schwinger, "Quantized gravitational field," Phys. Rev. **130** (1963) 1253.
[7] L. D. Faddeev, "The energy problem in Einstein's theory of gravitation," Sov. Phys. Usp. **25** (1982) 130 [Usp. Fiz. Nauk **136** (1982) 435].

St.Petersburg Department of Steklov Mathematical Institute