

## SCATTERING THEORY FOR A THREE-PARTICLE SYSTEM

L. D. FADDEEV

Mathematics Institute, Academy of Sciences, U.S.S.R.

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It is shown that the eigenfunctions of the Hamiltonian of a three-particle system with pair interaction can be represented in a natural fashion as the sum of three terms, for each of which there exists a linked set of equations. This set can be written down with the help of the solutions of the pair problems, and can be solved uniquely, in contrast to the equations of the Lippman-Schwinger type. In the limit of zero range, the well-known Skornyakov-Ter-Martirosyan equations are obtained.

At the present time, the basic premises of scattering theory, with redistribution of the particles, have been worked out in as much detail as in the theory of elastic scattering. To be precise, non-stationary (Ekstein<sup>1</sup>) and the stationary (Gerjuoy<sup>2</sup>) statements of the scattering problem have been given, a unique procedure has been formulated for the writing down of the element of the S matrix for any possible process, and the asymptotic value of the eigenfunctions of the Hamiltonian of the system in configuration space has been expressed in terms of these matrix elements. However, integral equations of the Lippman-Schwinger type,<sup>3,4</sup> which are easily derived for these functions even in this general case,<sup>4,5</sup> possess a number of inadequacies. In the first place, these equations possess solutions for both the homogeneous and inhomogeneous equation, so that the solution is not determined uniquely by the free term. In the second place, it is difficult in these equations to proceed to the case in which the pair interactions are specified not by means of potentials, but, for example, by means of boundary conditions (solid spheres) or by the specifying of the logarithmic derivative of the solution for  $r = 0$  (zero-range forces).

We obtain below for the eigenfunctions equations that are free of these inadequacies. It is shown that the eigenfunctions can be represented in a natural way as sums of three terms, for which there exists a linked set of equations. In the case of three particles with pair interaction, there are three such terms and all the equations are inhomogeneous, while the homogeneous equation has a solution only for energies corresponding to the bound state of the entire system. To determine the kernels of the integral equations it is neces-

sary to solve only the pair problem. These kernels are generalizations of the so-called T matrix, and can easily be constructed for different limiting cases, in which there is no potential. We note that the natural division of the wave function in the three particle problem into three terms has appeared in several researches devoted to the question (see, for example, references 7 and 8).

The derivation of the equations is given in the first section. We begin by recalling well-known premises of scattering theory. In the second section, the equations are written down in the momentum representation. The meaning of pair kernels is explained there in detail. The equations are written in terms of these kernels. In the third section, a number of remarks are made in connection with the equations that have been derived.

## 1. FORMAL THEORY

The method will be illustrated by an example of three nonrelativistic spinless particles with masses  $m_1, m_2, m_3$ . The particles are assumed to be different. The Hamiltonian of the system has the form ( $\hbar = 1$ )

$$H = T_1 + T_2 + T_3 + V_{23} + V_{31} + V_{12}, \quad (1)$$

$$T_i = - (1/2m_i) \nabla_i^2, \quad i = 1, 2, 3; \quad (2)$$

$V_{ij}$  acts only on the variable  $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$  of the wave function and vanishes as  $r_{ij} \rightarrow \infty$ .

In this system, infinite motions are possible, whether of all three particles or of the bound state of any pair plus the third particle. We shall prove that each pair can form bound states. This situation is expressed in the asymptotic behavior of the wave functions of the system. The asymptotic basis functions  $\Phi_n$  are generally separated into

four classes. The functions

$$\Phi_{n_0} = \exp \{ik_1 r_1 + ik_2 r_2 + ik_3 r_3\} \quad (3)$$

with energies

$$E_{n_0} = k_1^2/2m_1 + k_2^2/2m_2 + k_3^2/2m_3 \quad (4)$$

describe the free motion of all three particles.

The free motion of the bound state of particles 2 and 3 and of particle 1 is described by the functions

$$\Phi_{n_{23}} = \exp \{ik_1 r_1 + iK_{23} R_{23}\} \varphi_{23}^{(l)}(r_{23}), \quad (5)$$

$$R_{23} = (m_2 r_2 + m_3 r_3)/(m_2 + m_3), \quad r_{23} = r_2 - r_3. \quad (6)$$

Here  $\varphi_{23}^{(l)}(r)$  is the solution of the equation

$$[-(1/2\mu_{23})\nabla_r^2 + v_{23}(r)]\varphi_{23}^{(l)}(r) = -\lambda_{23}^{(l)}\varphi_{23}^{(l)}(r), \quad (7)$$

where

$$\mu_{23} = m_2 m_3 / (m_2 + m_3), \quad \lambda_{23}^{(l)} \geq 0, \quad l = 1, 2, \dots,$$

i.e., the function that describes the bound state. The corresponding energy is determined by the formula

$$E_{n_{23}} = k_1^2/2m_1 + K_{23}^2/2(m_2 + m_3) - \lambda_{23}^{(l)}. \quad (8)$$

The functions  $\Phi_{n_{31}}$  and  $\Phi_{n_{12}}$  can be constructed similarly.

The element of the S matrix that characterizes the transition from the state described asymptotically by the function  $\Phi_i$  to the state described by the function  $\Phi_f$  is determined by the expression

$$R_{fi} = \langle \Phi_f, V_f \Psi_i \rangle, \quad E_i = E_f. \quad (9)$$

Here  $V_f$  is the part of the interaction which is not taken into account in the state  $\Phi_f$ , while the function  $\Psi_i$  is determined by means of the resolvent

$$G(z) = (H - z)^{-1} \quad (10)$$

according to the formula

$$\Psi_n = -i\epsilon G(E_n + i\epsilon) \Phi_n. \quad (11)$$

The functions  $\Psi_n$  are the eigenfunctions of the total Hamiltonian H with eigenvalues  $E_n$ . Equations of the Lippman-Schwinger type for these functions are easily constructed from the corresponding equation for the resolvent G(z):

$$G(z) = G_0(z) - G_0(z)VG(z), \quad (12)$$

$$G_0(z) = (H_0 - z)^{-1},$$

$$H_0 = T_1 + T_2 + T_3, \quad V = H - H_0. \quad (13)$$

Multiplying both sides of (12) by  $-i\epsilon$  and applying the equation to  $\Phi_n$ , we get

$$\Psi_n = -i\epsilon G_0(E_n + i\epsilon) \Phi_n - G_0(E_n + i0)V\Psi_n. \quad (14)$$

It is not difficult to see that

$$-i\epsilon G_0(E_{n_0} + i\epsilon) \Phi_{n_0} = \Phi_{n_0}, \quad (15)$$

so that we get an equation of the Lippman-Schwinger type for  $\Psi_{n_0}$ :

$$\Psi_{n_0} = \Phi_{n_0} - G_0(E_{n_0} + i0)V\Psi_{n_0}. \quad (16)$$

Here,

$$-i\epsilon G_0(E_{n_{23}} + i\epsilon) \Phi_{n_{23}} = 0, \quad (17)$$

since

$$\begin{aligned} G_0(z) \Phi_{n_{23}} &= \exp \{ik_1 r_1 + iK_{23} R_{23}\} \\ &\times \frac{2\mu_{23}}{4\pi} \int \frac{\exp \{i[2\mu_{23}(z - k_1^2/2m_1 - K_{23}^2/2(m_2 + m_3))^{1/2} |r_{23} - r'_{23}|]\}}{|r_{23} - r'_{23}|} \\ &\times \varphi_{23}^{(l)}(r'_{23}) d^3 r'_{23} \end{aligned} \quad (18)$$

does not possess a singularity for  $z = E_{n_{23}}$ . Thus  $\Psi_{n_{23}}$  is the solution of the homogeneous equation in terms of the Green's function  $G_0$ :

$$\Psi_{n_{23}} = -G_0(E_{n_{23}} + i0)V\Psi_{n_{23}}. \quad (19)$$

If we use the Green's function

$$G_{23}(z) = (H_0 + V_{23} - z)^{-1}, \quad (20)$$

then we get inhomogeneous equations for  $\Psi_{n_0}$  and  $\Psi_{n_{23}}$ . However, the equations for  $\Psi_{n_{31}}$  and  $\Psi_{n_{12}}$ , expressed in terms of this Green's function, will be homogeneous.

Actually we have, for example,

$$\begin{aligned} G_{23}(z) \Phi_{n_{31}} &= \exp \{iK R\} \int d^3 p \exp \{i p p_1\} \left[ \left( \frac{1}{2\pi} \right)^3 \int \varphi_{31}(r) \right. \\ &\times \exp \left\{ -ir \left( p + \frac{m_1}{m_1 + m_3} p_2 \right) \right\} d^3 r \left. \right] \\ &\times \left[ \int g_{23}(r_{23}, r', z - \frac{p^2}{2\mu_1} - \frac{K^2}{2M}) \right. \\ &\times \exp \left\{ -ir' \left( \frac{m_2}{m_2 + m_3} p + p_2 \right) \right\} d^3 r' \left. \right]. \end{aligned} \quad (21)$$

Here  $g_{23}(r, r', z)$  is the solution of the equation

$$[-(2\mu_{23})^{-1}\nabla_r^2 + v_{23}(r) - z]g_{23}(r, r', z) = \delta^3(r - r'), \quad (22)$$

which, as is known, has poles at  $z = -\lambda_{23}^{(l)}$ ,  $l = 1, 2, \dots$ ;

$$R = M^{-1}(m_1 r_1 + m_2 r_2 + m_3 r_3),$$

$$p_1 = (m_2 + m_3)^{-1}(m_2 r_2 + m_3 r_3) - r_1, \quad K = K_{31} + k_2,$$

$$\mu_1 = m_1(m_2 + m_3)/M, \quad p_2 = M^{-1}(m_2 K_{31} - (m_1 + m_3)k_2),$$

$$M = m_1 + m_2 + m_3. \quad (23)$$

After integration over p we get a smooth function of z, such that

$$-i\epsilon G_{23}(E_{n_{31}} + i\epsilon)\Phi_{n_{31}} = 0. \quad (24)$$

Thus, in the case under examination, equations of the Lippman-Schwinger type do not have unique solutions, and the choice of the latter must be carried out through additional requirements, for example, asymptotic behavior in all directions in configuration space. But this means that the equations do not satisfy the fundamental role of integral equations — that of combining the differential equation and the boundary conditions in a single relation. Therefore, these equations are ill-suited for finding the solutions in practice.

We shall begin the derivation of our equations for the eigenfunctions by revising Eq. (12) for  $G(z)$ . We shall seek  $G(z)$  in the form

$$G(z) = G_0(z) - G_0(z)T(z)G_0(z). \quad (25)$$

We have for  $T(z)$  the equation

$$T(z) = V - VG_0(z)T(z). \quad (26)$$

We now consider the successive approximations for this equation:

$$T(z) = V_{23} + V_{31} + V_{12}$$

$$- (V_{23} + V_{31} + V_{12})G_0(z)(V_{23} + V_{31} + V_{12}) + \dots \quad (27)$$

and reconstruct the series in the following way:

$$\begin{aligned} T(z) = & V_{23} - V_{23}G_0(z)V_{23} + V_{23}G_0(z)V_{23}G_0(z)V_{23} - \dots \\ & + V_{31} - V_{31}G_0(z)V_{31} + \dots + V_{12} - V_{12}G_0(z)V_{12} + \dots \\ & + (V_{23} - V_{23}G_0(z)V_{23} + \dots)G_0(z) \\ & \times (V_{31} - V_{31}G_0(z)V_{31} + \dots) + \dots \end{aligned} \quad (28)$$

We denote by  $T_{23}(z)$  the sum of the chain

$$T_{23}(z) = V_{23} - V_{23}G_0(z)V_{23} + \dots \quad (29)$$

Obviously,  $T_{23}(z)$  is the solution of the equation

$$T_{23}(z) = V_{23} - V_{23}G_0(z)T_{23}(z), \quad (30)$$

which is obtained from (26) if we set the two potentials  $V_{31}$  and  $V_{12}$  equal to zero. In Eq. (30) the variables are separated trivially (this is done below in detail) and the operator  $T_{23}(z)$  is characterized only by the pair interaction of particles 2 and 3.  $T_{31}(z)$  and  $T_{12}(z)$  are introduced similarly. It is not difficult to write down the series for (27) by means of these operators. It suffices to replace all  $V_{ij}$  in (27) by the corresponding  $T_{ij}(z)$  and to omit terms that contain a factor of the type  $T_{23}(z)G_0(z)T_{23}(z)$ , i.e., where  $T_{ij}(z)$  with the same indices appear side by side.

It is not possible to write down the equation for which the resultant series would be the successive-approximation series. However, if we represent  $T(z)$  as a sum of three terms

$$T(z) = T^{(1)}(z) + T^{(2)}(z) + T^{(3)}(z), \quad (31)$$

where, for example,

$$T^{(1)}(z) = V_{23} - V_{23}G_0(z)T(z) \quad (32)$$

and  $T^{(2)}(z)$  and  $T^{(3)}(z)$  are similarly defined, then these components satisfy a linked set of equations, which are conveniently written in matrix form:

$$\begin{pmatrix} T^{(1)}(z) \\ T^{(2)}(z) \\ T^{(3)}(z) \end{pmatrix} = \begin{pmatrix} T_{23}(z) \\ T_{31}(z) \\ T_{12}(z) \end{pmatrix} - \begin{pmatrix} 0 & T_{23}(z) & T_{23}(z) \\ T_{31}(z) & 0 & T_{31}(z) \\ T_{12}(z) & T_{12}(z) & 0 \end{pmatrix} G_0(z) \begin{pmatrix} T^{(1)}(z) \\ T^{(2)}(z) \\ T^{(3)}(z) \end{pmatrix}. \quad (33)$$

It is not difficult to verify that the successive approximations for this equation duplicate the series described above. Naturally, the validity of the resultant system does not depend on the convergence of this series. A derivation of Eq. (33) without the use of perturbation theory will be published elsewhere.

We now obtain the corresponding equations for the functions  $\Psi_n$ . In accord with the expansion (31), we have

$$G(z) = G_0(z) + G^{(1)}(z) + G^{(2)}(z) + G^{(3)}(z), \quad (34)$$

where the functions

$$G^{(i)}(z) = -G_0(z)T^{(i)}G_0(z), \quad i = 1, 2, 3, \quad (35)$$

satisfy the equation

$$\begin{pmatrix} G^{(1)} \\ G^{(2)} \\ G^{(3)} \end{pmatrix} = \begin{pmatrix} G_{23} - G_0 \\ G_{31} - G_0 \\ G_{12} - G_0 \end{pmatrix} - G_0 \begin{pmatrix} 0 & T_{23} & T_{23} \\ T_{31} & 0 & T_{31} \\ T_{12} & T_{12} & 0 \end{pmatrix} \begin{pmatrix} G^{(1)} \\ G^{(2)} \\ G^{(3)} \end{pmatrix}. \quad (36)$$

Here the argument  $z$  is everywhere omitted,

$$G_{ij}(z) = (H_0 + V_{ij} - z)^{-1}. \quad (37)$$

Applying both sides of this equation (multiplied by  $-i\epsilon$ ) to the functions  $\Phi_{n_0}$  and  $\Phi_{n_{23}}$ , and taking into account equalities of the type (17) and (24), we get equations of the following form for  $\Psi_{n_0}$  and  $\Psi_{n_{23}}$ :

$$\Psi_{n_0} = \Phi_{n_0} + \Psi_{n_0}^{(1)} + \Psi_{n_0}^{(2)} + \Psi_{n_0}^{(3)} \quad (z = E_{n_0} + i0), \quad (38)$$

$$\begin{pmatrix} \Psi^{(1)} \\ \Psi^{(2)} \\ \Psi^{(3)} \end{pmatrix} = \begin{pmatrix} \Phi_{n_0}^{(23)} - \Phi_{n_0} \\ \Phi_{n_0}^{(31)} - \Phi_{n_0} \\ \Phi_{n_0}^{(12)} - \Phi_{n_0} \end{pmatrix} - G_0(z) \begin{pmatrix} 0 & T_{23}(z) & T_{23}(z) \\ T_{31}(z) & 0 & T_{31}(z) \\ T_{12}(z) & T_{12}(z) & 0 \end{pmatrix} \begin{pmatrix} \Psi^{(1)} \\ \Psi^{(2)} \\ \Psi^{(3)} \end{pmatrix}; \quad (39)$$

$$\Psi_{n_{23}} = \Psi_{n_{23}}^{(1)} + \Psi_{n_{23}}^{(2)} + \Psi_{n_{23}}^{(3)} \quad (z = E_{n_{23}} + i0), \quad (40)$$

$$\begin{pmatrix} \Psi^{(1)} \\ \Psi^{(2)} \\ \Psi^{(3)} \end{pmatrix} = \begin{pmatrix} \Phi_{n_{23}} \\ 0 \\ 0 \end{pmatrix} - G_0(z) \begin{pmatrix} 0 & T_{23}(z) & T_{23}(z) \\ T_{31}(z) & 0 & T_{31}(z) \\ T_{12}(z) & T_{12}(z) & 0 \end{pmatrix} \begin{pmatrix} \Psi^{(1)} \\ \Psi^{(2)} \\ \Psi^{(3)} \end{pmatrix}. \quad (41)$$

Here, in order to avoid encumbering the formulas,

we have omitted at many places a number of indices characterizing the corresponding functions.

The equations for  $\Psi_{n_{31}}$  and  $\Psi_{n_{12}}$  are similar to the equation for  $\Psi_{n_{23}}$  except that the free term differs from zero in the second and third place, respectively. In Eqs. (39) we denote by  $\Phi_{n_0}^{(ij)}$  the limits

$$\Phi_{n_0}^{(ij)} = -i\varepsilon G_{ij}(E_{n_0} + i\varepsilon) \Phi_{n_0}. \quad (42)$$

It is not difficult to establish that, for example,

$$\Phi_{n_0}^{(23)} = \exp\{ik_1 r_1 + iK_{23} R_{23}\} \varphi_{23}(r_{23}, k_{23}), \quad (43)$$

where  $\varphi_{23}(\mathbf{r}, \mathbf{k})$  is a solution of the equation

$$[-(2\mu_{23})^{-1}\nabla_r^2 + v_{23}(\mathbf{r})] \varphi_{23}(\mathbf{r}, \mathbf{k}) = k^2 \varphi_{23}(\mathbf{r}, \mathbf{k}), \quad (44)$$

such that  $\varphi_{23}(\mathbf{r}, \mathbf{k}) - e^{i\mathbf{k}\cdot\mathbf{r}}$  is a diverging wave in which

$$\mathbf{K}_{23} = \mathbf{k}_2 + \mathbf{k}_3, \quad \mathbf{k}_{23} = (m_2 \mathbf{k}_2 - m_3 \mathbf{k}_3)/(m_2 + m_3). \quad (45)$$

## 2. MOMENTUM REPRESENTATION

The momentum representation is especially suitable for writing down these equations. We can introduce three different systems of coordinates in place of the coordinates  $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$  in this representation. For example, in the set

$$\begin{aligned} \mathbf{K} &= \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3, & \mathbf{k}_{23} &= (m_3 \mathbf{k}_2 - m_2 \mathbf{k}_3)/(m_2 + m_3), \\ \mathbf{p}_1 &= M^{-1}[m_1(\mathbf{k}_2 + \mathbf{k}_3) - (m_2 + m_3)\mathbf{k}_1], & M &= m_1 + m_2 + m_3 \end{aligned} \quad (46)$$

the kernel of the operator  $T_{23}(z)$  assumes an especially simple form:

$$\begin{aligned} T_{23}(k_1, k_2, k_3, k'_1, k'_2, k'_3; z) \\ = \delta^3(\mathbf{K} - \mathbf{K}') \delta^3(\mathbf{p}_1 - \mathbf{p}'_1) t_{23}(k_{23}, k'_{23}; z - K^2/2M \\ - p_1^2/2\mu_1), \end{aligned} \quad (47)$$

where  $\mu_1 = m_1(m_2 + m_3)/M$ , and  $t_{23}(\mathbf{k}, \mathbf{k}'; z)$  is a solution of an equation of the type

$$t(\mathbf{k}, \mathbf{k}'; z) = v(\mathbf{k} - \mathbf{k}') - \int \frac{v(\mathbf{k} - \mathbf{k}'') t(\mathbf{k}'', \mathbf{k}'; z)}{k''^2/2\mu - z} d^3 k''. \quad (48)$$

In operator form, (48) has the form

$$t(z) = v - v g_0(z) t(z), \quad (49)$$

$$g_0(z) = (h_0 - z)^{-1}, \quad h_0 = -(1/2\mu) \nabla^2. \quad (50)$$

We see that  $t(\mathbf{k}, \mathbf{k}'; z)$  represents a kernel of the pair T matrix, where all the arguments have independent values. The scattering amplitude of particles 2 and 3 on one another is proportional in their center-of-mass system to  $t_{23}(\mathbf{k}, \mathbf{k}'; z)$  for  $2\mu_{23}z = k^2 + i0 = k'^2 + i0$ .

It is convenient to write down the kernels of the operators  $T_{31}(z)$  and  $T_{12}(z)$  in the coordinates

$\mathbf{K}, \mathbf{k}_{31}, \mathbf{p}_2$  and  $\mathbf{K}, \mathbf{k}_{12}, \mathbf{p}_3$ , respectively. Expressions for these coordinates and for the kernels of the operators  $T_{31}(z)$  and  $T_{12}(z)$  are obtained by interchanging of the indices in the given formulas.

The operator  $G_0(z)$  is the operator of multiplication by the function

$$\begin{aligned} \left(\frac{K^2}{2M} + \frac{k_{23}^2}{2\mu_{23}} + \frac{p_1^2}{2\mu_1} - z\right)^{-1} &= \left(\frac{K^2}{2M} + \frac{k_{31}^2}{2\mu_{31}} + \frac{p_2^2}{2\mu_2} - z\right)^{-1} \\ &= \left(\frac{K^2}{2M} + \frac{k_{12}^2}{2\mu_{12}} + \frac{p_3^2}{2\mu_3} - z\right)^{-1}. \end{aligned} \quad (51)$$

All the  $\Phi_{\mathbf{n}}$  functions contain the factor  $\delta^3(\mathbf{K} - \mathbf{K}_0)$ , where  $\mathbf{K}_0$  is the total momentum of the system.\*

It is not difficult to transform to the center-of-mass system, dropping this factor and introducing a new reference point for the energy

$$\bar{E}_n = E_n - K_0^2/2M. \quad (52)$$

Below we shall operate in the center-of-mass system and will not write the bar over  $E_{\mathbf{n}}$ .

Our system becomes especially simple if we introduce separate coordinates for each component  $\Psi^{(i)}$ :

$$\begin{aligned} \Psi^{(1)} &= \Psi^{(1)}(\mathbf{k}_{23}, \mathbf{p}_1), & \Psi^{(2)} &= \Psi^{(2)}(\mathbf{k}_{31}, \mathbf{p}_2), \\ \Psi^{(3)} &= \Psi^{(3)}(\mathbf{k}_{12}, \mathbf{p}_3). \end{aligned} \quad (53)$$

For these functions, the equation take on the following form:

$$\begin{aligned} \bar{\Psi}_n(\mathbf{k}, \mathbf{p}) &= \bar{\Phi}_n(\mathbf{k}, \mathbf{p}) - \int A(\mathbf{k}, \mathbf{p}, \mathbf{k}', \mathbf{p}'; \\ z = E_n + i0) \bar{\Psi}_n(\mathbf{k}', \mathbf{p}') d^3 k' d^3 p'. \end{aligned} \quad (54)$$

Here  $\bar{\Psi}_{\mathbf{n}}$  and  $\bar{\Phi}_{\mathbf{n}}$  are columns of three components, while A is a three-row square matrix. The elements  $a^{(ij)}$  of this matrix have the form

$$\begin{aligned} a^{(11)} &= a^{(22)} = a^{(33)} = 0, \\ a^{(12)} &= \left(\frac{k^2}{2\mu_{23}} + \frac{p^2}{2\mu_1} - z\right)^{-1} t_{23}\left(\mathbf{k}, -\frac{m_2}{m_2 + m_3} \mathbf{p} - \mathbf{p}', z - \frac{p^2}{2\mu_1}\right) \\ &\quad \times \delta^3\left(\mathbf{k}' - \mathbf{p} - \frac{m_1}{m_1 + m_3} \mathbf{p}'\right), \\ a^{(13)} &= \left(\frac{k^2}{2\mu_{23}} + \frac{p^2}{2\mu_1} - z\right)^{-1} t_{23}\left(\mathbf{k}, \frac{m_3}{m_2 + m_3} \mathbf{p} + \mathbf{p}'; z - \frac{p^2}{2\mu_1}\right) \\ &\quad \times \delta^3\left(\mathbf{k}' + \mathbf{p} + \frac{m_1}{m_1 + m_2} \mathbf{p}'\right). \end{aligned} \quad (55)$$

The expressions for  $a^{(23)}$  and  $a^{(31)}$  or for  $a^{(21)}$  and  $a^{(32)}$  are obtained from expressions for  $a^{(12)}$  or  $a^{(13)}$  by the corresponding interchange of the indices in the right hand sides of (55).

Equations (54) and (55) are the final forms of the equations used. The potentials  $V_{ij}$  do not ap-

\*Here and throughout we denote the quantum numbers by an upper or lower index 0 to distinguish them from the arguments of the wave functions.

pear explicitly in these equations. The mathematical solvability of these equations will be treated elsewhere.

### 3. DISCUSSION OF THE DERIVED EQUATIONS

1. The equations become simplified in the case of identical particles. Thus, as an example, all the  $\Psi^{(i)}(\mathbf{k}, \mathbf{p})$  coincide for a symmetric eigenfunction  $\Psi_{\text{sym}}$ , so that we replace the set by the single equation

$$\begin{aligned} \Psi_n(\mathbf{k}, \mathbf{p}) &= \Phi_n(\mathbf{k}, \mathbf{p}) \\ &- \frac{m}{k^2 + 3p^2/4 - mz} \int \left[ t\left(\mathbf{k}, -\frac{1}{2}\mathbf{p} - \mathbf{p}'; z - \frac{3}{4}\frac{p^2}{m}\right) \right. \\ &\times \Psi_n\left(\mathbf{p} + \frac{1}{2}\mathbf{p}', \mathbf{p}'\right) + t\left(\mathbf{k}, \frac{1}{2}\mathbf{p} + \mathbf{p}'; z - \frac{3}{4}\frac{p^2}{m}\right) \\ &\times \Psi_n\left(-\mathbf{p} - \frac{1}{2}\mathbf{p}', \mathbf{p}'\right) \left. \right] d^3p', \\ z &= E_n + i0. \end{aligned} \quad (56)$$

If  $\Psi_{\text{sym}}$  corresponds to scattering by a bound system, then

$$\Phi_n(\mathbf{k}, \mathbf{p}) = \delta^3(\mathbf{p} - \mathbf{p}_0) \varphi(\mathbf{k}), \quad E_n = 3p_0^2/4 - \lambda, \quad (57)$$

where  $\varphi(\mathbf{k})$  is the function describing the bound state. By solution of Eq. (56),  $\Psi_{\text{sym}}$  is expressed in the form

$$\begin{aligned} \Psi_{\text{sym}} &= \Psi_n(\mathbf{k}_{23}, \mathbf{p}_1) + \Psi_n(\mathbf{k}_{31}, \mathbf{p}_2) + \Psi_n(\mathbf{k}_{12}, \mathbf{p}_3) \\ &= 3 \text{sym}_{\mathbf{k}, \mathbf{p}} [\Psi(\mathbf{k}_{23}, \mathbf{p}_1)]. \end{aligned} \quad (58)$$

The symbol sym denotes symmetry with respect to the variables written beneath it.

If  $\Psi_{\text{sym}}$  corresponds to free motion of all three particles, then

$$\begin{aligned} \Phi_n(\mathbf{k}, \mathbf{p}) &= m \frac{t(\mathbf{k}, \mathbf{k}_0; k_0^2/m + i0)}{k^2 - k_0^2 - i0} \delta^3(\mathbf{p} - \mathbf{p}_0), \\ E_n &= \frac{k_0^2}{m} + \frac{3}{4} \frac{p_0^2}{m}, \end{aligned} \quad (59)$$

$$\begin{aligned} \Psi_{\text{sym}} &= \text{sym}_{\mathbf{k}, \mathbf{p}; \mathbf{k}^0, \mathbf{p}^0} [\delta^3(\mathbf{k}_{23} - \mathbf{k}_{23}^0) \delta^3(\mathbf{p}_1 - \mathbf{p}_1^0) \\ &+ 3\Psi_n(\mathbf{k}_{23}, \mathbf{p}_1; \mathbf{k}_{23}^0, \mathbf{p}_1^0)]. \end{aligned} \quad (60)$$

In the approximation of zero range of forces,  $t(\mathbf{k}, \mathbf{k}'; z)$  does depend on neither  $\mathbf{k}$  nor  $\mathbf{k}'$ :

$$t(\mathbf{k}, \mathbf{k}'; z) = [(2\pi)^2 \mu (\alpha + i\sqrt{2\mu z})]^{-1}, \quad \text{Im}\sqrt{z} \geq 0. \quad (61)$$

Here  $\alpha$  is the inverse scattering length. If  $\alpha > 0$ , then the pair problem has a bound state, in which the corresponding  $\varphi(\mathbf{k})$  has the form

$$\varphi(\mathbf{k}) = N/(k^2 + \alpha^2), \quad (62)$$

where  $N$  is a normalizing factor. Equation (56) shows that in the corresponding problem with three

particles, the symmetric function for scattering on a bound state is determined by Eq. (58), where  $\Psi_n(\mathbf{k}, \mathbf{p})$  has the form

$$\Psi_n(\mathbf{k}, \mathbf{p}) = \frac{Nm}{k^2 + 3p^2/4 - mz} \left( \delta^3(\mathbf{p} - \mathbf{p}_0) - \frac{1}{\pi^2} \frac{\varphi(\mathbf{p}, \mathbf{p}_0)}{\alpha + i\sqrt{mz - 3p^2/4}} \right), \quad (63)$$

where the following equation is valid for  $\varphi(\mathbf{p}, \mathbf{p}_0)$ :

$$\begin{aligned} \varphi(\mathbf{p}, \mathbf{p}_0) &= (p^2 + \mathbf{p}\mathbf{p}_0 + p_0^2 - mz)^{-1} \\ &- \frac{1}{\pi^2} \int \frac{\varphi(\mathbf{p}', \mathbf{p}_0) d^3p'}{(p^2 + \mathbf{p}\mathbf{p}' + p'^2 - mz)(\alpha + i\sqrt{mz - 3p'^2/4})}. \end{aligned} \quad (64)$$

This equation is identical with the equation of Skorniyakov-Ter-Martirosyan.<sup>9</sup> This is not difficult to prove by making the substitution

$$\varphi(\mathbf{p}, \mathbf{p}_0) = \frac{3}{8} \frac{a(\mathbf{p}, \mathbf{p}_0)}{\alpha - i\sqrt{mz - 3p^2/4}}. \quad (65)$$

2. The potentials can be entirely eliminated from the expressions for the matrix elements of the  $S$  matrix. We shall carry out the corresponding derivation for the case of a process in which particle 3 is scattered from a bound state of particles 1 and 2 to produce a bound state of particles 3 and 1 and a free particle 2, i.e., for the process

$$(1, 2) + 3 \rightarrow 2 + (1, 3). \quad (66)$$

The matrix element is determined by the expression

$$R_{fi} = \langle \Phi_f, V_f \Psi_i \rangle, \quad E_f = E_i, \quad (67)$$

where

$$\begin{aligned} \Phi_f &= \Phi_{n_{31}}, \quad V_f = V_{23} + V_{12}, \\ E_f &= 2\mu_2 (p_2^0)^2 - \lambda_{31}, \quad E_i = 2\mu_3 (p_3^0)^2 - \lambda_{12}, \\ \Psi_i &= \Psi_{n_{12}} = \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)}. \end{aligned} \quad (68)$$

Here  $\lambda_{31}$  and  $\lambda_{12}$  are the corresponding positive binding energies.

With the help of an equation of the type (30) for  $T_{23}$  and (41) for  $\Psi_{n_{12}}$ , we get

$$\begin{aligned} V_{23} \Psi^{(1)} &= -V_{23} G_0 T_{23} (\Psi^{(2)} + \Psi^{(3)}) = -V_{23} (\Psi^{(2)} \\ &+ \Psi^{(3)}) + T_{23} (\Psi^{(2)} + \Psi^{(3)}) \end{aligned} \quad (69)$$

and, consequently,

$$V_{23} \Psi_i = T_{23} (\Psi^{(2)} + \Psi^{(3)}). \quad (70)$$

Similarly,

$$V_{12} \Psi_i = V_{12} \Phi_{n_{12}} + T_{12} (\Psi^{(1)} + \Psi^{(2)}) \quad (71)$$

and thus

$$\begin{aligned} \langle \Phi_f, V_f \Psi_i \rangle &= \langle \Phi_{n_{31}}, V_{12} \Phi_{n_{12}} \rangle \\ &+ \langle \Phi_{n_{31}}, T_{23}(z) (\Psi^{(2)} + \Psi^{(3)}) + T_{12}(z) (\Psi^{(1)} + \Psi^{(2)}) \rangle, \\ z &= E_i + i0. \end{aligned} \quad (72)$$

The potential  $V_{12}$  can also be eliminated from the first term. Simple calculation with the aid of the equation for  $\varphi_{12}(\mathbf{r})$  shows that

$$\langle \Phi_{n_1}, V_{12} \Phi_{n_2} \rangle = - \left[ \frac{1}{2\mu_{12}} \left( \mathbf{p}_2^0 + \frac{m_2}{m_1 + m_2} \mathbf{p}_3^0 \right)^2 + \lambda_{12} \right] \times \varphi_{12} \left( \mathbf{p}_2^0 + \frac{m_2}{m_1 + m_2} \mathbf{p}_3^0 \right) \varphi_{31} \left( \mathbf{p}_3^0 + \frac{m_3}{m_1 + m_3} \mathbf{p}_2^0 \right), \quad (73)$$

where  $\varphi_{12}(\mathbf{k})$  and  $\varphi_{31}(\mathbf{k})$  are functions describing the bound states in the momentum representation.

The matrix elements for other processes can be transformed in a similar fashion. A special paper will be devoted to various similar formulas, to the proof of the completeness of the eigenfunctions and the unitarity of the S matrix, and to other premises of the so-called formal scattering theory.

3. Obviously, the re-formulation of the equations of scattering theory given here can be carried out in the case of an arbitrary number of particles  $n \geq 3$ . If there are only pair interactions, then the number of equations in the system will be  $N = n(n-1)/2$ . These equations will be useful if only there are no more bound states than pairs of particles, since otherwise solutions of the homogeneous equations will exist. If triple or more complicated bound states exist, then a more complex construction must be used for  $n \geq 4$ . In this case, it is natural to assume the existence of triple or more complicated interactions.

In any case, for pair interactions in which repulsion predominates, equations similar to those obtained can be shown to be useful. An especially great simplification is obtained for the problem with identical particles. In this case, the system reduces to a single equation. We note that in the

nonstationary formulation (more precisely, in the problem of statistical physics, where the inverse temperature plays a role similar to the time) the reconstruction of a series similar to (27) has been carried out by Yang and Lee.<sup>10</sup> However, these authors did not write out any integral equation, of which there is one in our case, since the particles are considered to be identical. It is possible that this equation can be useful for the problems of statistical physics.

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